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Some applications of Noether's theorem

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ABSTRACT

If the action $S = \int_{t_1}^{t_2} L(q,\dot{q},t)\, dt$ is invariant under the infinitesimal transformation $\tilde{t} = t + \varepsilon\, \tau(q,t)$, $\tilde{q}_r = q_r + \varepsilon\, \xi_r(q,t)$, $r = 1, \dots, n$, with $\varepsilon = \text{constant} \ll 1$, then the Noether's theorem permits to construct the corresponding conserved quantity. The Lanczos approach employs to $\varepsilon = q_{n+1}$ as a new degree of freedom, thus the Euler-Lagrange equation for this new variable gives the Noether's constant of motion. Torres del Castillo and Rubalcava-García showed that each variational symmetry implies the existence of an ignorable coordinate; here we apply the Lanczos approach to the Noether's theorem to motivate the principal relations of these authors. The Maxwell equations without sources are invariant under duality rotations, then we show that this invariance implies, via the Noether's theorem, the continuity equation for the electromagnetic energy. Besides, we demonstrate that if we know one solution of $p(x)y'' + q(x)y' + r(x)y = 0$, then this Lanczos technique allows obtain the other solution of this homogeneous linear differential equation.

***Keywords*:** Noether's theorem, Invariance of the action, Lanczos variational method, Ignorable variable, Variational symmetry, Duality rotations, Maxwell equations, Complex Riemann-Silberstein vector, Linear differential equation of second order, Variation of parameters

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1. INTRODUCTION

In the functional (the concept of action was proposed by Leibnitz [1]) $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$ we apply the infinitesimal transformation (ε = constant $\ll 1$):

$$\tilde{t} = t + \varepsilon\, \tau(q, t), \qquad \tilde{q}_r = q_r + \varepsilon\, \xi_r(q, t), \quad r = 1, \dots, n \tag{1}$$

that is

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L\left(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t}\right) d\tilde{t}, \tag{2}$$

then we say that the action is invariant if:

$$\tilde{S} = S + \varepsilon \int_{t_1}^{t_2} \frac{d}{dt} Q(q, t)\, dt, \tag{3}$$

hence the Euler-Lagrange equations (Lagrangian expressions [2, 3]) corresponding to the variational principle $\delta S = 0$:

$$E_r \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1, \dots, n \tag{4}$$

remain intact. Noether [2] studied the case $Q = 0$, and she suggested [3, 4] to Bessel-Hagen [5] the analysis of (4) with $Q \neq 0$ [6].

Therefore, we have a symmetry up to divergence and Noether [2, 5-9] proved the existence of the Rund-Trautman identity [7, 8, 10, 11]:

$$\frac{\partial L}{\partial q_r}\, \xi_r + \frac{\partial L}{\partial \dot{q}_r}\, \dot{\xi}_r + \frac{\partial L}{\partial t}\, \tau - \left(\frac{\partial L}{\partial \dot{q}_r}\, \dot{q}_r - L\right) \dot{\tau} - \frac{dQ}{dt} = 0, \tag{5}$$

which can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\, \xi_r - H\tau - Q\right) = (\xi_r - \dot{q}_r \tau) E_r\,, \qquad H = \frac{\partial L}{\partial \dot{q}_c}\, \dot{q}_c - L\,. \tag{6}$$

In (5) and (6) we use the convention of Dedekind [12, 13]-Einstein because we sum over repeated indices. The Rund-Trautman identity offers a more efficient test of invariance [8]. If in (6) we employ the Euler-Lagrange equations (4) we deduce the constant of motion associated to (1):

$$\varphi(q, \dot{q}, t) \equiv \frac{\partial L}{\partial \dot{q}_r}\, \xi_r - H\tau - Q = Constant, \tag{7}$$

hence we have a connection between symmetries and conservation laws [3, 7, 8, 10, 14-16].

We remember the following words of Havas [3, 4]: ‘The relation between symmetries and conserved quantities could fail for physical systems whose equations could not be written in Hamiltonian form’. On the importance of the symmetries we have the comment of Weinberg [17]: 'Wigner realized, earlier than most physicists, the outstanding of thinking about symmetries as objects of interest in themselves, quite apart from the dynamical theory. The symmetries of Nature are the deepest things we know about it'.

In Sec. 2 we exhibit the Lanczos technique [18-24] to obtain the Noether's conserved quantity (7) as the Euler-Lagrange equation for the parameter $\varepsilon(t)$. The Sec. 3 contains a motivation for the result of Torres del Castillo and Rubalcava-García [25], namely, that a variational symmetry implies the existence of an ignorable coordinate. In Sec. 4 we show that the continuity equation for the electromagnetic energy can be deduced from the invariance of Maxwell equations under duality rotations [21]. The Sec. 5 has an application of the Noether's theorem to an arbitrary homogeneous linear differential equation of second order [26].

2. LANCZOS APPROACH TO CONSERVATION LAWS

Lanczos [18, 19] applies the infinitesimal transformation (1) (with $\varepsilon =$ constant) to the action (2) and uses expansion of Taylor up to first order in ε, thus:

$$\tilde{S} = S + \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_r}\, \xi_r + \frac{\partial L}{\partial \dot{q}_r}\, \dot{\xi}_r + \frac{\partial L}{\partial t}\, \tau - H\dot{\tau}\right) dt, \tag{8}$$

hence this integrand is equal to $\frac{dQ}{dt}$, in harmony with the Rund-Trautman identity (5).

Now Lanczos proposes to employ (1) into (2) but considering that ε is a function, therefore up to 1th order in ε:

$$\tilde{S} = \int_{t_1}^{t_2} \left[L + \varepsilon\, \frac{dQ}{dt} + \dot{\varepsilon}\left(\frac{\partial L}{\partial \dot{q}_r}\, \xi_r - H\tau\right)\right] dt = \int_{t_1}^{t_2} \bar{L}\; dt, \tag{9}$$

and he accepts that ε is a new degree of freedom with its corresponding Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\varepsilon}}\right) - \frac{\partial \bar{L}}{\partial \varepsilon} = 0. \tag{10}$$

It is clear that:

$$\frac{\partial \bar{L}}{\partial \dot{\varepsilon}} = \frac{\partial L}{\partial \dot{q}_r}\, \xi_r - H\tau, \qquad \frac{\partial \bar{L}}{\partial \varepsilon} = \frac{dQ}{dt},$$

therefore (10) implies (7). In other words, if the parameter of the symmetry is considered as an additional degree of freedom of the variational principle, then its Euler-Lagrange equation gives the Noether's constant of motion. We comment that Neuenschwander [27] obtains the conserved quantity (7) for the case $Q = 0$. If in (8) we use τ and ξ_r as new degrees of freedom, then the corresponding Euler-Lagrange equations imply the equations of motion (4) and the known relation $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$. The works [20, 22] have applications of this Lanczos technique to some singular Lagrangians employed in [9, 28-30]. In [18, 21] and [31] the Lanczos method is applied to electromagnetic and gravitational fields, respectively.

3. IGNORABLE VARIABLES AND CONSTANTS OF MOTION

Here we look a finite transformation of coordinates:

$$t\ ,\ q_r \qquad \longrightarrow \qquad t',\ q'_r\ , \qquad r = 1, \dots, n \tag{11}$$

such that in the new Lagrangian one coordinate, we say q'_1 , participates as ignorable variable and its conjugate momentum leads to the constant (7); Torres del Castillo and Rubalcava-García [25] show that (11) can be obtained from the equations:

$$\frac{\partial t}{\partial q'_1} = \tau\ , \qquad \frac{\partial q_r}{\partial q'_1} = \xi_r\ , \qquad r = 1, \dots, n\ . \tag{12}$$

The Lanczos variational technique [18, 19] allows deduce the Noether's conserved quantity (7) as the Euler-Lagrange equation for the parameter ε if it is considered as a new degree of freedom. Here we employ this Lanczos approach to motivate the relations (12).

Lanczos applies (1) into (2) but considering that ε is a function, therefore up to 1th order in ε:

$$\tilde{S} = \int_{t_1}^{t_2} \left[L + \dot{\varepsilon}\,\varphi(q,\dot{q},t) + \frac{d}{dt}(\varepsilon Q)\right] dt = \int_{t_1}^{t_2} \left[\bar{L} + \frac{d}{dt}(\varepsilon Q)\right]\ dt, \tag{13}$$

where we use (5) and (7); thus we can see that $\varepsilon(t)$ is ignorable into $\bar{L}$ and its corresponding Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\varepsilon}}\right) - \frac{\partial \bar{L}}{\partial \varepsilon} = 0$ implies that φ is a constant.

On the other hand, from (1) we have that $t = \tilde{t} - \varepsilon\,\tau(q,t)$ and $q_r = \tilde{q}_r - \varepsilon\,\xi_r(q,t)$, then:

$$\frac{\partial t}{\partial \varepsilon} = -\,\tau\ , \qquad \frac{\partial q_r}{\partial \varepsilon} = -\,\xi_r\ , \qquad r = 1, \dots, n \tag{14}$$

but ε is ignorable and its momentum leads to (7), hence it is natural the identification $\varepsilon = -\,q'_1$, thus (14) imply the expressions (12) obtained by Torres del Castillo & Rubalcava-García [25], and in their paper we find several examples on the construction of ignorable variables associated to variational symmetries [32]. Let's remember [18] that the fundamental importance of the ignorable variables for the integration of the Lagrangian equations was first recognized by Routh [33] and Helmholtz [34].

4. MAXWELL EQUATIONS AND DUALITY ROTATIONS

The Maxwell equations in absence of sources are given by [18] $\vec{\nabla}\cdot\vec{\mathrm{E}} = 0,\ \ \vec{\nabla}\cdot\vec{\mathrm{B}} = 0$ and:

$$\vec{\nabla}\times\vec{\mathrm{E}} + \frac{\partial \vec{\mathrm{B}}}{\partial t} = \vec{0}, \qquad \vec{\nabla}\times\vec{\mathrm{B}} - \frac{1}{\mathrm{c}^2}\frac{\partial \vec{\mathrm{E}}}{\partial t} = \vec{0}, \tag{15}$$

where $\vec{\mathrm{B}}$ and $\vec{\mathrm{E}}$ are the magnetic and electric fields, respectively; $c = 1/\sqrt{\varepsilon_0 \mu_0}$ denotes the light velocity in empty space. From (15) it is immediate the conservation law of the electromagnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\mathrm{E}^2 + \frac{1}{2\mu_0}\mathrm{B}^2\right) + \vec{\nabla}\cdot\vec{\mathrm{P}} = 0, \qquad \vec{\mathrm{P}} = \frac{1}{\mu_0}\vec{\mathrm{E}}\times\vec{\mathrm{B}}, \tag{16}$$

with $\vec{\mathrm{P}}$ the Poynting vector [35].

The equations (15) are invariant under the duality rotations [21, 36-42]:

$$c\vec{\mathrm{B}}' = c\vec{\mathrm{B}}\cos\alpha - \vec{\mathrm{E}}\sin\alpha, \qquad \vec{\mathrm{E}}' = c\vec{\mathrm{B}}\sin\alpha + \vec{\mathrm{E}}\cos\alpha, \tag{17}$$

because the fields (17) also satisfy (15). Here we employ the Lanczos approach to Noether theorem to show that the invariance of (15) under (17) implies (16). In fact, if we use the complex Riemann-Silberstein vector [18, 42-46] $\vec{\mathrm{F}} = c\vec{\mathrm{B}} + i\,\vec{\mathrm{E}}$, $i = \sqrt{-1}$, then the relations (15) are equivalent to:

$$\vec{\nabla}\times\vec{\mathrm{F}} + \frac{i}{c}\frac{\partial\vec{\mathrm{F}}}{\partial t} = \vec{0}, \tag{18}$$

and the duality rotations (17) represent the change of phase:

$$\vec{\mathrm{F}}' = e^{i\alpha}\vec{\mathrm{F}}, \qquad \alpha = constant. \tag{19}$$

On the other hand, the Maxwell equations (18) can be deduced from the principle [18]:

$$\delta\int_{V_4}\mathrm{L}\,d^4x = 0, \qquad \mathrm{L} = \vec{\mathrm{F}}^*\cdot\left(\vec{\nabla}\times\vec{\mathrm{F}} + \frac{i}{c}\frac{\partial\vec{\mathrm{F}}}{\partial t}\right), \qquad \vec{\mathrm{F}}^* = c\vec{\mathrm{B}} - i\,\vec{\mathrm{E}}, \tag{20}$$

where $\vec{\mathrm{F}}$ and $\vec{\mathrm{F}}^*$ are the variational variables. It is clear that L is invariant under the global symmetry (19) with α = constant, then the Lanczos approach to Noether's theorem indicates to calculate L′ but now with (19) as an infinitesimal local symmetry because $\alpha(\vec{r},t) \ll 1$ is a new degree of freedom, with its corresponding Euler-Lagrange equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial\mathrm{L}'}{\partial\alpha_{,x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\mathrm{L}'}{\partial\alpha_{,y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\mathrm{L}'}{\partial\alpha_{,z}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial\mathrm{L}'}{\partial\alpha_{,t}}\right) = 0. \tag{21}$$

Therefore:

$$\mathrm{L}' \equiv \vec{\mathrm{F}}'^*\cdot\left(\vec{\nabla}\times\vec{\mathrm{F}}' + \frac{i}{c}\frac{\partial\vec{\mathrm{F}}'}{\partial t}\right), \qquad \vec{\mathrm{F}}' = e^{i\alpha(\vec{r},t)}\vec{\mathrm{F}} = (1+i\alpha)\vec{\mathrm{F}}, \qquad \alpha \ll 1,$$

$$= \mathrm{L} - i\left(\vec{\mathrm{F}}^*\times\vec{\mathrm{F}}\right)\cdot\vec{\nabla}\alpha - \frac{1}{c}\vec{\mathrm{F}}^*\cdot\vec{\mathrm{F}}\,\alpha_{,t}\,, \qquad \text{to first order in } \alpha,$$

thus (21) implies the continuity equation:

$$\frac{\partial}{\partial t}\left(\vec{\mathrm{F}}^{*}\cdot\vec{\mathrm{F}}\right)+i\,\vec{\nabla}\cdot\left(c\,\vec{\mathrm{F}}^{*}\mathrm{x}\,\vec{\mathrm{F}}\right)=0, \tag{22}$$

which is equivalent to (16) because $\vec{\mathrm{F}}^{*}\cdot\vec{\mathrm{F}}=c^{2}\mathrm{B}^{2}+\mathrm{E}^{2}$ and $c\,\vec{\mathrm{F}}^{*}\,\mathrm{x}\,\vec{\mathrm{F}}=-\frac{2i}{\varepsilon_0}\vec{\mathrm{P}}$. The process here realized manifests that, in the source-free case, the invariance of Maxwell equations under duality rotations is the underlying symmetry into the conservation law of the electromagnetic energy [21].

5. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

Now we consider the differential equation [26]:

$$p(x)\,y''+\,q(x)\,y'+\,r(x)\,y=\,0, \tag{23}$$

and we accept to have the solution $y_1(x)$:

$$p\,y_1{}''+\,q\,y_1'+\,r\,y_1=\,0. \tag{24}$$

It is very known [48] that the Abel-Liouville-Ostrogradski expression [49] for the wronskian:

$$W\equiv y_1\,y_2'-y_2\,y_1'=\exp\left(-\int^{x}\frac{q(\xi)}{p(\xi)}\,d\xi\right), \tag{25}$$

allows to obtain other solution for (23):

$$y_2(x)=\,y_1(x)\,\int^{x}\frac{W(\xi)}{[y_1(\xi)]^2}\,d\xi. \tag{26}$$

Here we employ a variational approach to construct (26); in fact, it is easy to see that the action [50]:

$$S=\int_{x_1}^{x_2}L(y,y',x)\,dx=\int_{x_1}^{x_2}\left(y'^2-\frac{r}{p}y^2\right)\exp\left(\int^{x}\frac{q}{p}d\xi\right)dx, \tag{27}$$

implies (23) under the condition $\delta S=0$, that is, the Euler-Lagrange expression [18] $\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)-\frac{\partial L}{\partial y}=0$ leads to our homogeneous differential equation. Now we show that (27) admits a variational symmetry [32], hence we apply the Noether's theorem in the Lanczos approach to prove that the corresponding conserved quantity gives the relation (26) for the solution $y_2(x)$. In fact, we use y_1 to introduce the infinitesimal transformation:

$$\tilde{y}(x)=y(x)+\varepsilon\,y_1(x),\qquad \varepsilon\ll 1, \tag{28}$$

where ε is a constant parameter, then the new Lagrangian into (27) is given by:

$$\tilde{L} = (\tilde{y}'^2 - \tfrac{r}{p}\,\tilde{y}^2)\exp(\textstyle\int^x \tfrac{q}{p}\,d\xi),$$
$$\overset{(28)}{=} L + \frac{d}{dx}\left[2\varepsilon\, y\, y_1' \exp\left(\int^x \frac{q}{p}\,d\xi\right)\right] - \frac{2\varepsilon}{p} y\,(p y_1'' + q y_1' + r y_1)\exp\left(\int^x \frac{q}{p}\,d\xi\right),$$
$$\overset{(24)}{=} L + \frac{d}{dx}\left[2\varepsilon\, y\, y_1' \exp\left(\int^x \frac{q}{p}\,d\xi\right)\right], \tag{29}$$

therefore the Lagrangian is invariant up to divergence, hence (28) is a variational symmetry of the action (27).

To study the constant of motion associated to this symmetry (28) we apply the Noether theorem via the Lanczos technique, that is, into L we employ the transformation (28) but now $\varepsilon(x)$ is a new degree of freedom:

$$\tilde{L} = L + 2\left[\left(y' y_1' - \frac{r}{p} y\, y_1\right)\varepsilon + y' y_1\, \varepsilon'\right]\exp\left(\int^x \frac{q}{p}\,d\xi\right), \tag{30}$$

thus the Euler-Lagrange equation $\frac{d}{dx}\left(\frac{\partial \tilde{L}}{\partial \varepsilon'}\right) - \frac{\partial \tilde{L}}{\partial \varepsilon} = 0$ and (24) imply the relation:

$$\frac{d}{dx}\left[(y' y_1 - y\, y_1')\exp\left(\int^x \frac{q}{p}\,d\xi\right)\right] = 0, \tag{31}$$

whose integration gives the solution (26). Thus, the Lanczos version of Noether theorem shows that the structure of (26) is consequence from a variational symmetry of the action associated to the homogeneous differential equation (23).

We can study the general solution of second order linear differential equation:

$$p(x)y'' + q(x)y' + r(x)y = \phi(x), \tag{32}$$

via an alternative (but equivalent) method to the variation of parameters technique of Newton (Principia)-Bernoulli-Euler-Lagrange [48, 51-53]. We have the Abel-Liouville-Ostrogradski expression (25) for the wronskian where y_1 and y_2 are solutions of the corresponding homogeneous equation (23):

$$p(x){y''}_k + q(x){y'}_k + r(x)y_k = 0, \qquad k = 1, 2\,, \tag{33}$$

hence if we know y_1 then from (25) we can to construct the solution (26).

Now we show a procedure to obtain the particular solution y_p verifying:

$$p(x){y''}_p + q(x){y'}_p + r(x)y_p = \phi(x), \tag{34}$$

and it is not necessary the Lagrange's ansatz; thus the general solution of (32) is given by:

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) + y_p(x). \tag{35}$$

If we multiply (33), for $k = 1$, by $\frac{1}{Wp}$ and we use (25), it is easy to deduce the relation:

$$r\,\frac{y_1}{Wp} = -\frac{d}{dx}\left(\frac{y_1'}{W}\right). \tag{36}$$

Similarly, if we multiply (32) by $\frac{\lambda}{p}$, where $\lambda(x)$ is a function to determine, we find that:

$$\left[\frac{r\lambda}{p} - \frac{d}{dx}\left(\frac{\lambda q}{p} - \lambda'\right)\right] y + \frac{d}{dx}\left[\left(\frac{\lambda q}{p} - \lambda'\right) y + \lambda y'\right] = \frac{\lambda}{p}\,\phi, \tag{37}$$

whose left side is an exact derivative if we ask:

$$r\,\frac{\lambda}{p} = \frac{d}{dx}\left(\frac{\lambda q}{p} - \lambda'\right) \equiv -\frac{d}{dx}\left[\frac{1}{W}(W\lambda)'\right], \tag{38}$$

then its comparison with (36) leads to $\lambda = \frac{y_1}{W}$, and (37) acquires the form:

$$\frac{d}{dx}\left[\frac{{y_1}^2}{W}\,\frac{d}{dx}\left(\frac{y}{y_1}\right)\right] = \frac{y_1 \phi}{pW}, \tag{39}$$

therefore, two successive integrations of (39) imply the general solution (35) with y_2 given by (26), and the particular solution:

$$y_p(x) = y_2(x) \int^x \frac{y_1(\xi)\phi(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x) \int^x \frac{y_2(\xi)\phi(\xi)}{p(\xi)W(\xi)}\,d\xi\,, \tag{40}$$

in harmony with the variation of parameters method [26, 48, 54]; we consider that our approach justifies the traditional Lagrange's ansatz employed in that method.

We note that the action (27) can be extended to [55]:

$$\bar{S} = \int_{x_1}^{x_2} \left({y'}^2 - \frac{r}{p} y^2 + \frac{2\phi}{p}\,y\right) \exp\left(\int^x \frac{q(\xi)}{p(\xi)}\,d\xi\right)\,dx, \tag{41}$$

such that $\delta\bar{S} = 0$ implies (32).

6. CONCLUSIONS

The energy-momentum tensor of the electromagnetic field $\mathrm{T}^{\mu\nu} = \mathrm{T}^{\nu\mu}$, whose symmetry was required by Planck [56] to secure the relativistic equivalence between mass and energy, has null trace $(\mathrm{T}^{\mu}{}_{\mu} = 0)$ because the photon being massless; the photon has spin 1 because the Maxwell spinor possesses two indices [57]. Thus, we have the following quadratic expression in the Faraday tensor [58]:

$$\mathrm{T}^{\mu\nu} = -\mathrm{F}^{\mu\alpha}\,\mathrm{F}^{\nu}{}_{\alpha} + \frac{1}{4}\,\mathrm{I}_1\,g^{\mu\nu}\,, \qquad \mathrm{I}_1 = \mathrm{F}^{\alpha\beta}\mathrm{F}_{\alpha\beta}\,, \tag{42}$$

which is invariant under duality rotations as expected [that is, (42) is not altered if we use the fields (17)], due to the fact that this symmetry is associated with the conservation of the electromagnetic energy. Warwick [59] exposes an interesting story about how Poynting discovered the continuity equation (16). In Sec. 5 we considered the linear differential equation of second order and we showed that our process (without the ansatz of Lagrange) motivates the method of variation of parameters; it must be valuable apply our approach, based in the Lanczos version of Noether's theorem, to differential equations of third and fourth order [26].

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