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SHORT COMMUNICATION

On the Faddeev-Sominsky's algorithm

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ABSTRACT

We comment that the Faddeev-Sominsky's process to obtain an inverse matrix is equivalent to the Cayley-Hamilton-Frobenius theorem plus the Leverrier-Takeno's method to construct the characteristic polynomial of an arbitrary matrix. Besides, we deduce the Lanczos expression for the resolvent of the corresponding matrix.

***Keywords*:** Inverse matrix, Characteristic equation, Eigenvalue problem, Adjoint matrix, Faddeev-Sominsky's method, Leverrier-Takeno's algorithm, Resolvent of a matrix

1. INTRODUCTION

For an arbitrary matrix $\mathbf{A}_{nxn} = (\mathrm{A}^{i}{}_{j})$ its characteristic equation [1-3]:

$$p(\lambda) \equiv \lambda^n + a_1\,\lambda^{n-1} + \cdots + a_{n-1}\,\lambda + a_n = 0, \tag{1}$$

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can be obtained, through several procedures [1, 4-7], directly from the condition det $(A^i{}_j - \lambda\, \delta^i_j) = 0$. The approach of Leverrier-Takeno [4, 8-12] is a simple and interesting technique to construct (1) based in the traces of the powers $\mathbf{A}^r,\ r = 1, \dots, n$.

On the other hand, it is well known that an arbitrary matrix satisfies (1), which is the Cayley-Hamilton-Frobenius identity [1-3]:

$$\mathbf{A}^n + a_1\, \mathbf{A}^{n-1} + \cdots + a_{n-1}\, \mathbf{A} + a_n\, \mathbf{I} = \mathbf{0}. \tag{2}$$

If **A** is non-singular (that is, det $\mathbf{A} \neq 0$), then from (2) we obtain its inverse matrix:

$$\mathbf{A}^{-1} = -\frac{1}{a_n}(\mathbf{A}^{n-1} + a_1\, \mathbf{A}^{n-2} + \cdots + a_{n-1}\, \mathbf{I}), \tag{3}$$

where $a_n \neq 0$ because $a_n = (-1)^n \det \mathbf{A}$.

Faddeev-Sominsky [13-15] proposed an algorithm to determine $\mathbf{A}^{-1}$ in terms of $\mathbf{A}^r$ and their traces, which is equivalent [16] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix **A**; we also show the Lanczos expression for the resolvent of **A** , that is, the Laplace transform of $\exp(t\,\mathbf{A})$, to see Sec. 2.

2. LEVERRIER-TAKENO AND FADDEEV-SOMINSKY TECHNIQUES

If we define the quantities:

$$a_0 = 1, \qquad s_k = tr\, \mathbf{A}^k, \qquad k = 1, 2, \dots, n \tag{4}$$

then the process of Leverrier-Takeno [4, 8-12] implies (1) wherein the a_i are determined with the Newton's recurrence relation:

$$r\, a_r + s_1\, a_{r-1} + s_2\, a_{r-2} + \ \dots + s_{r-1}\, a_1 + s_r = 0, \qquad r = 1, 2, \dots, n \tag{5}$$

therefore:

$$\begin{gathered} a_1 = -s_1\,, \qquad 2!\ a_2 = (s_1)^2 - s_2\,, \qquad 3!\ a_3 = -(s_1)^3 + 3\, s_1\, s_2 - 2\, s_3\,, \\ 4!\ a_4 = (s_1)^4 - 6\,(s_1)^2\, s_2 + 8\, s_1\, s_3 + 3\,(s_2)^2 - 6\, s_4\,, \quad etc. \end{gathered} \tag{6}$$

in particular, det $\mathbf{A} = (-1)^n\, a_n$, that is, the determinant of any matrix only depends on the traces s_r , which means that **A** and its transpose have the same determinant. In [17, 18] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & \cdots & 0 \\ s_2 & s_1 & k-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_{k-2} & \cdots & \cdots & 1 \\ s_k & s_{k-1} & \cdots & \cdots & s_1 \end{vmatrix}, \qquad k = 1, \dots, n\,. \tag{7}$$

The Faddeev-Sominsky's procedure [13-16, 19, 20] to obtain $\mathbf{A}^{-1}$ is a sequence of algebraic computations on the powers $\mathbf{A}^r$ and their traces, in fact, this algorithm is given via the instructions:

$$\begin{array}{lll} \mathbf{B}_1 = \mathbf{A}, & q_1 = \mathrm{tr}\,\mathbf{B}_1, & \mathbf{C}_1 = \mathbf{B}_1 - q_1\,\mathbf{I}, \\ \mathbf{B}_2 = \mathbf{C}_1\,\mathbf{A}, & q_2 = \frac{1}{2}\,\mathrm{tr}\,\mathbf{B}_2, & \mathbf{C}_2 = \mathbf{B}_2 - q_2\,\mathbf{I}, \\ \vdots & \vdots & \vdots \\ \mathbf{B}_{n-1} = \mathbf{C}_{n-2}\,\mathbf{A}, & q_{n-1} = \frac{1}{n-1}\,\mathrm{tr}\,\mathbf{B}_{n-1}, & \mathbf{C}_{n-1} = \mathbf{B}_{n-1} - q_{n-1}\,\mathbf{I}, \\ \mathbf{B}_n = \mathbf{C}_{n-1}\,\mathbf{A}, & q_n = \frac{1}{n}\,\mathrm{tr}\,\mathbf{B}_n, & \end{array} \tag{8}$$

then:

$$\mathbf{A}^{-1} = \frac{1}{q_n}\,\mathbf{C}_{n-1}. \tag{9}$$

For example, if we apply (8) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (9) reproduces (3). By mathematical induction one can prove that (8) and (9) are equivalent to (3), (4) and (5), showing [16] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (8) we can see that [20]:

$$\mathbf{C}_k = \mathbf{A}^k + a_1\mathbf{A}^{k-1} + a_2\mathbf{A}^{k-2} + \ldots + a_{k-1}\mathbf{A} + a_k\mathbf{I}, \quad k = 1, 2, \ldots, n-1,$$

$$\mathbf{C}_n = \mathbf{B}_n - q_n\mathbf{I} = \mathbf{0}, \tag{10}$$

and for $k = n - 1$:

$$\mathbf{C}_{n-1} = \mathbf{A}^{n-1} + a_1\,\mathbf{A}^{n-2} + a_2\,\mathbf{A}^{n-3} + \ldots + a_{n-2}\,\mathbf{A} + a_{n-1}\,\mathbf{I} \overset{(3)}{=} -a_n\,\mathbf{A}^{-1},$$

in harmony with (9) because $a_n = -q_n$. The property $\mathbf{C}_n = \mathbf{0}$ is equivalent to (2); if $\mathbf{A}$ is singular, the process (8) gives the adjoint matrix of $\mathbf{A}$ [2, 3, 14], in fact, $Adj\,\mathbf{A} = (-1)^{n+1}\,\mathbf{C}_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of $\mathbf{A}$ [6]:

$$\mathbf{A}\,\vec{u}_k = \lambda_k\,\vec{u}_k, \qquad k = 1, 2, \ldots, n, \tag{11}$$

because for a given value of k, each column of:

$$\mathbf{Q}_k \equiv \lambda_k^{n-1}\,\mathbf{I} + \lambda_k^{n-2}\,\mathbf{C}_1 + \cdots + \mathbf{C}_{n-1}\,, \tag{12}$$

satisfies (11) [14, 21], and therefore all columns of $\mathbf{Q}_k$ are proportional to each other, that is, $rank\ \mathbf{Q}_k = 1$ [19].

Now we consider the matrix:

$$\mathbf{Q}(z) \equiv z^{n-1}\,\mathbf{I} + z^{n-2}\,\mathbf{C}_1 + z^{n-3}\,\mathbf{C}_2 + \cdots + z\,\mathbf{C}_{n-2} + \mathbf{C}_{n-1}\,, \quad \mathbf{Q}(\lambda_k) = \mathbf{Q}_k\,, \tag{13}$$

then from (8):

$$\mathbf{Q}(z) = z^{n-1}\,\mathbf{I} + z^{n-2}\,(\mathbf{B}_1 + a_1\,\mathbf{I}) + z^{n-3}\,(\mathbf{B}_2 + a_2\,\mathbf{I}) + \cdots + z\,(\mathbf{B}_{n-2} + a_{n-2}\,\mathbf{I}) + \mathbf{B}_{n-1} + a_{n-1}\,\mathbf{I}\,,$$

$$= (z^{n-1} + a_1\,z^{n-2} + a_2\,z^{n-3} + \cdots + a_{n-2}\,z + a_{n-1})\,\mathbf{I} + (z^{n-2}\,\mathbf{I} + z^{n-3}\,\mathbf{C}_1 + \cdots + z\,\mathbf{C}_{n-3} + \mathbf{C}_{n-2})\,\mathbf{A}\,,$$

$$\overset{(1),(13)}{=} \frac{1}{z}\,[\,p(z) - a_n\,]\,\mathbf{I} + \frac{1}{z}\,[\,\mathbf{Q}(z) - \mathbf{C}_{n-1}\,]\,\mathbf{A} \overset{(8)}{=} \frac{1}{z}\,[\,p(z) + \mathbf{Q}(z)\,\mathbf{A}\,] - [\,a_n\,\mathbf{I} + \mathbf{B}_n\,], \tag{14}$$

but from (10) we have the relation $\mathbf{B}_n + a_n\,\mathbf{I} = \mathbf{0}$, therefore (14) implies the following Lanczos formula [20-25] for the resolvent of $\mathbf{A}$:

$$\frac{1}{z\,\mathbf{I} - \mathbf{A}} = \frac{\mathbf{Q}(z)}{p(z)}\,. \tag{15}$$

If $\mathbf{A}$ is non-singular, then (15) for $z = 0$ implies (9). McCarthy [26] used (15) and the Cauchy's integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2). We note that (15) is the Laplace transform of $\exp(t\,\mathbf{A})$ [25].

If the roots of (1) have distinct values, then from (15) the Faddeev-Sominsky's method allows construct the proper vectors of $\mathbf{A}$, in fact, they are given via the expression [14, 21]:

$$\mathbf{A}\,\mathbf{Q}_k = \lambda_k\,\mathbf{Q}_k, \qquad \mathbf{Q}_k \neq \mathbf{0}, \qquad k = 1, \ldots, n, \tag{16}$$

in according with (11).

The coefficients a_k defined in (6) and (7) can be written in terms of the Bell polynomials [27-33], in fact [34]:

$$a_m = \frac{1}{m!}\,Y_m(-0!\ s_1,\ -1!\ s_2,\ -2!\ s_3,\ -3!\ s_4,\ \ldots\ ,\ -(m-2)!\ s_{m-1},\ -(m-1)!\ s_m). \tag{17}$$

On the other hand, Sylvester [35-38] obtained the following interpolating definition of $f(\mathbf{A})$:

$$f(\mathbf{A}) = \sum_{j=1}^{n} f(\lambda_j)\,\prod_{k\neq j} \frac{\mathbf{A}-\lambda_k\,\mathbf{I}}{\lambda_j-\lambda_k}, \tag{18}$$

which is valid if all eigenvalues are different from each other. Buchheim [39] generalized (18) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. We comment that with (18) also is possible to prove the relation (15) for the resolvent of **A**.

3. CONCLUSIONS

It is interesting to mention that the method (8) was successfully applied [40] in general relativity to study the embedding of spacetimes into pseudo-Euclidean spaces. The Leverrier-Takeno and Faddeev-Sominsky methods need $\frac{1}{2}(n-1)(2\,n^3-2\,n^2+n+2)$ and $(n-1)\,n^3$ arithmetic operations to determine the coefficients a_k, respectively. Gower [19] indicates that the procedure (8) is subject to unacceptable rounding errors and therefore is unsuitable for numerical work, hence such algorithm does have algebraic interest.

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