SHORT COMMUNICATION

On the Faddeev-Sominsky’s algorithm

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ABSTRACT

We comment that the Faddeev-Sominsky’s process to obtain an inverse matrix is equivalent to the Cayley-Hamilton-Frobenius theorem plus the Leverrier-Takeno’s method to construct the characteristic polynomial of an arbitrary matrix. Besides, we deduce the Lanczos expression for the resolvent of the corresponding matrix.

Keywords: Inverse matrix, Characteristic equation, Eigenvalue problem, Adjoint matrix, Faddeev-Sominsky’s method, Leverrier-Takeno’s algorithm, Resolvent of a matrix

1. INTRODUCTION

For an arbitrary matrix \( A_{nxn} = (A^T_j) \) its characteristic equation [1-3]:

\[
p(\lambda) \equiv \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n = 0,
\]

(Received 27 July 2018; Accepted 06 August 2018; Date of Publication 07 August 2018)
can be obtained, through several procedures [1, 4-7], directly from the condition \( \det (A^r - \lambda I) = 0 \). The approach of Leverrier-Takeno [4, 8-12] is a simple and interesting technique to construct (1) based in the traces of the powers \( A^r, \ r = 1, ..., n \).

On the other hand, it is well known that an arbitrary matrix satisfies (1), which is the Cayley-Hamilton-Frobenius identity [1-3]:

\[
A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0. \tag{2}
\]

If \( A \) is non-singular (that is, \( \det A \neq 0 \)), then from (2) we obtain its inverse matrix:

\[
A^{-1} = -\frac{1}{a_n} (A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I),
\tag{3}
\]

where \( a_n \neq 0 \) because \( a_n = (-1)^n \det A \).

Faddeev-Sominsky [13-15] proposed an algorithm to determine \( A^{-1} \) in terms of \( A^r \) and their traces, which is equivalent [16] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno’s method to construct the characteristic polynomial of a matrix \( A \); we also show the Lanczos expression for the resolvent of \( A \), that is, the Laplace transform of \( \exp(t A) \), to see Sec. 2.

2. LEVERRIER-TAKENO AND FADDEEV-SOMINSKY TECHNIQUES

If we define the quantities:

\[
a_0 = 1, \quad s_k = tr A^k, \quad k = 1, 2, ..., n \tag{4}
\]

then the process of Leverrier-Takeno [4, 8-12] implies (1) wherein the \( a_i \) are determined with the Newton’s recurrence relation:

\[
a_r s_1 a_{r-1} + s_2 a_{r-2} + \cdots + s_{r-1} a_1 + s_r = 0, \quad r = 1, 2, ..., n \tag{5}
\]

therefore:

\[
a_1 = -s_1, \quad 2! \ a_2 = (s_1)^2 - s_2, \quad 3! \ a_3 = -(s_1)^3 + 3 s_1 s_2 - 2 s_3,
\]

\[
4! \ a_4 = (s_1)^4 - 6 \ (s_1)^2 s_2 + 8 s_1 s_3 + 3 \ (s_2)^2 - 6 s_4, \quad etc. \tag{6}
\]

in particular, \( \det A = (-1)^n a_n \), that is, the determinant of any matrix only depends on the traces \( s_r \), which means that \( A \) and its transpose have the same determinant. In [17, 18] we find the general expression:

\[
a_k = \frac{(-1)^k}{k!} \begin{vmatrix}
\begin{array}{cccc}
s_1 & k-1 & 0 & \cdots & 0 \\
s_2 & s_1 & k-2 & \cdots & 0 \\
s_3 & \vdots & \ddots & \ddots & \vdots \\
s_{k-1} & s_{k-2} & \ddots & \ddots & 1 \\
s_k & s_{k-1} & \cdots & \cdots & s_1
\end{array}
\end{vmatrix}, \quad k = 1, ..., n. \tag{7}
\]
The Faddeev-Sominsky’s procedure [13-16, 19, 20] to obtain $A^{-1}$ is a sequence of algebraic computations on the powers $A^r$ and their traces, in fact, this algorithm is given via the instructions:

\[
\begin{align*}
B_1 &= A, & q_1 &= \text{tr} B_1, & C_1 &= B_1 - q_1 I, \\
B_2 &= C_1 A, & q_2 &= \frac{1}{2} \text{tr} B_2, & C_2 &= B_2 - q_2 I, \\
& \vdots & & \vdots & & \vdots \\
B_{n-1} &= C_{n-2} A, & q_{n-1} &= \frac{1}{n-1} \text{tr} B_{n-1}, & C_{n-1} &= B_{n-1} - q_{n-1} I, \\
B_n &= C_{n-1} A, & q_n &= \frac{1}{n} \text{tr} B_n,
\end{align*}
\]

(8)

then:

\[
A^{-1} = \frac{1}{q_n} C_{n-1}.
\]

(9)

For example, if we apply (8) for $n = 4$, then it is easy to see that the corresponding $q_r$ imply (6) with $q_j = -a_j$, and besides (9) reproduces (3). By mathematical induction one can prove that (8) and (9) are equivalent to (3), (4) and (5), showing [16] thus that the Faddeev-Sominsky’s technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (8) we can see that [20]:

\[
C_k = A^k + a_1 A^{k-1} + a_2 A^{k-2} + \ldots + a_{k-1} A + a_k I, \quad k = 1, 2, \ldots, n-1,
\]

(10)

and for $k = n-1$:

\[
C_{n-1} = A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-2} A + a_{n-1} I = -a_n A^{-1},
\]

in harmony with (9) because $a_n = -q_n$. The property $C_n = 0$ is equivalent to (2); if $A$ is singular, the process (8) gives the adjoint matrix of $A$ [2, 3, 14], in fact, $\text{Adj} A = (-1)^{n+1} C_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky’s algorithm allows obtain the corresponding eigenvectors of $A$ [6]:

\[
A \tilde{u}_k = \lambda_k \tilde{u}_k, \quad k = 1, 2, \ldots, n,
\]

(11)

because for a given value of $k$, each column of:
$$Q_k = \lambda_k^{n-1} I + \lambda_k^{n-2} C_1 + \cdots + C_{n-1}, \quad (12)$$
satisfies (11) [14, 21], and therefore all columns of $Q_k$ are proportional to each other, that is, $\text{rank } Q_k = 1$ [19].

Now we consider the matrix:

$$Q(z) \equiv z^{n-1} I + z^{n-2} C_1 + z^{n-3} C_2 + \cdots + z C_{n-2} + C_{n-1}, \quad Q(\lambda_k) = Q_k, \quad (13)$$

then from (8):

$$Q(z) = z^{n-1} I + z^{n-2} (B_1 + a_1 I) + z^{n-3} (B_2 + a_2 I) + \cdots + z (B_{n-2} + a_{n-2} I) + B_{n-1} + a_{n-1} I,$$

$$= (z^{n-1} + a_1 z^{n-2} + a_2 z^{n-3} + \cdots + a_{n-2} z + a_{n-1}) I$$
$$+ (z^{n-2} I + z^{n-3} C_1 + \cdots + z C_{n-3} + C_{n-2}) A,$$

(1), (13)

$$= \frac{1}{z} \left[ p(z) - a_n \right] I + \frac{1}{z} \left[ Q(z) - C_{n-1} \right] A = \frac{1}{z} \left[ p(z) + Q(z) A \right] - \left[ a_n I + B_n \right].$$

(8)

but from (10) we have the relation $B_n + a_n I = 0$, therefore (14) implies the following Lanczos formula [20-25] for the resolvent of $A$:

$$\frac{1}{z I - A} = \frac{Q(z)}{p(z)}. \quad (15)$$

If $A$ is non-singular, then (15) for $z = 0$ implies (9). McCarthy [26] used (15) and the Cauchy’s integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2). We note that (15) is the Laplace transform of $\exp(t A)$ [25].

If the roots of (1) have distinct values, then from (15) the Faddeev-Sominsky’s method allows construct the proper vectors of $A$, in fact, they are given via the expression [14, 21]:

$$A Q_k = \lambda_k Q_k, \quad Q_k \neq 0, \quad k = 1, \ldots, n, \quad (16)$$
in according with (11).

The coefficients $a_k$ defined in (6) and (7) can be written in terms of the Bell polynomials [27-33], in fact [34]:

$$a_m = \frac{1}{m!} Y_m (-0! s_1, -1! s_2, -2! s_3, -3! s_4, \ldots, -(m-2)! s_{m-1}, -(m-1)! s_m).$$

(17)

On the other hand, Sylvester [35-38] obtained the following interpolating definition of $f(A)$:

$$f(A) = \sum_{j=1}^n f(\lambda_j) \prod_{k \neq j} \frac{A - \lambda_k I}{\lambda_j - \lambda_k}, \quad (18)$$
which is valid if all eigenvalues are different from each other. Buchheim [39] generalized (18) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. We comment that with (18) also is possible to prove the relation (15) for the resolvent of $A$.

3. CONCLUSIONS

It is interesting to mention that the method (8) was successfully applied [40] in general relativity to study the embedding of spacetimes into pseudo-Euclidean spaces. The Leverrier-Takeno and Faddeev-Sominsky methods need $\frac{1}{2}(n - 1)(2n^3 - 2n^2 + n + 2)$ and $(n - 1)n^3$ arithmetic operations to determine the coefficients $a_k$, respectively. Gower [19] indicates that the procedure (8) is subject to unacceptable rounding errors and therefore is unsuitable for numerical work, hence such algorithm does have algebraic interest.

References


