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SHORT COMMUNICATION

Linear differential equations of second, third, and fourth order

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ABSTRACT

We study the linear differential equations of second, third, and fourth order, accepting to know solutions of the corresponding homogeneous equation, then we show how to obtain one more solution for the homogeneous case and a particular solution for the original equation.

***Keywords*:** Linear differential equation, Abel-Liouville-Ostrogradski identity, Euler-Lagrange method

1. INTRODUCTION

Given a linear differential equation of second order, the first step is to obtain one solution of the corresponding homogeneous equation (HE), after it is tradition to employ the Abel-Liouville-Ostrogradski identity to deduce another solution of this HE. Finally, the non-

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trivial 'ansatz' of Lagrange is applied to construct a particular solution of the original equation. Our technique uses self-adjoint and exact operators to solve this differential problem, it is not necessary the Lagrange's ansatz, in other words, our approach justifies the variation of parameters method. Besides, the process here exhibited gives, in natural manner, a simple deduction of the Lewis invariant associated to Ermakov-Milne-Pinney equation.

We accept that for the second order linear differential equation [1]:

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = \phi(x), \tag{1}$$

it is known the solution $y_1(x)$ of the corresponding HE:

$$p\, y'' + q\, y' + r\, y = 0, \tag{2}$$

then (2) admits the solution [2-5]:

$$y_2(x) = y_1(x) \int^x \frac{W(\eta)}{[y_1(\eta)]^2}\, d\eta, \tag{3}$$

with the expression of Abel-Liouville-Ostrogradski [3]:

$$W(x) = \exp\left(-\int^x \frac{q(\xi)}{p(\xi)}\, d\xi\right). \tag{4}$$

Besides, a particular solution of (1) is given by [2, 4, 6]:

$$y_p(x) = y_2(x) \int^x \frac{y_1(\xi)\,\phi(\xi)}{p(\xi)\,W(\xi)}\, d\xi - y_1(x) \int^x \frac{y_2(\xi)\,\phi(\xi)}{p(\xi)\,W(\xi)}\, d\xi\,, \tag{5}$$

obtained via the Variation of Parameters method [7-9] introduced by Newton (Principia), Bernoulli [10], Euler (1741) and Lagrange (1759, 1777).

In Sec. 2 we employ the concepts of exact and self-adjoint operators [11-14] to establish an alternative deduction of (3) and (5). The Sec. 3 shows a simple manner to construct the Lewis invariant [15-19] which is a constant of motion for the Ermakov [20]-Milne [21]-Pinney [22] equation. In Sec. 4 we extend the results of Sec. 2 to linear differential equations of third and fourth order [23, 24].

2. LINEAR DIFFERENTIAL OPERATOR

The functions $y_j(x)$ are solutions of (2), then $y_1(py_2'' + qy_2' + ry_2) - y_2(py_1'' + qy_1' + ry_1) = 0$, that is:

$$p\, \frac{dW}{dx} + q\, W = 0\,, \tag{6}$$

where: W is the wronskian [2-4, 25] of y_1 and y_2 :

$$W = y_1 \, y_2' - y_2 \, y_1'; \tag{7}$$

the integration of (6) and (7) gives (4) and (3), respectively.

The variation of parameters method uses the following ansatz to obtain a particular solution of (1):

$$y_p(x) = c_2(x) \, y_2(x) + c_1(x) \, y_1(x), \tag{8}$$

thus (1) implies:

$$p \, \frac{dA}{dx} + q \, A + p \, (c_1' \, y_1' + c_2' \, y_2') = \phi \, , \qquad A = c_1' \, y_1 + c_2' \, y_2 \, ,$$

where: we accept that $A = 0$, therefore $c_1' \, y_1 \, c_2' \, y_2 = 0$ and $c_1' \, y_1' + c_2' \, y_2' = \frac{\phi}{p}$, with determinant = $W \neq 0$, then $c_1' = \frac{y_2 \phi}{pW}$ & $c_2' = \frac{y_1 \phi}{pW}$, hence (5) is immediate from (8).

Now we shall employ the concepts of self-adjoint and exact operators [11-14] to exhibit an alternative approach to (3) and (5). In fact, (1) can be written in the form:

$$\hat{L} y = \phi \, , \qquad \hat{L} = p \, \frac{d^2}{dx^2} + q \, \frac{d}{dx} + r \, , \tag{9}$$

and it is known [11, 12] that $\gamma(x) \, \hat{L} = \hat{O}$ is a self-adjoint operator, $\hat{O}^\dagger = \hat{O}$, for:

$$\gamma(x) = \frac{1}{p \, W} = \frac{1}{p} \exp\left(\int^x \frac{q(\xi)}{p(\xi)} \, d\xi \right). \tag{10}$$

Besides, $\rho(x) \, \hat{O}$ is an exact operator [11]:

$$\frac{d}{dx}(\alpha \, y' + \beta \, y) = \rho \, \gamma \, \phi \, , \qquad \alpha = \rho \, \gamma \, p \, , \qquad \beta = \rho \, \gamma \, q - \alpha' \, , \tag{11}$$

if ρ satisfies the differential equation:

$$\gamma \, p \, \rho'' + [2 \, (\gamma \rho)' - \gamma \, q] \, \rho' + [(\gamma p)'' - (\gamma q)' + \gamma \, r] \, \rho = 0 \, ,$$

where we can apply (10) to obtain the following constraint on ρ :

$$p \, \rho'' + q \, \rho' + r \, \rho = 0 \, , \tag{12}$$

which coincides with the HE (2), then it is natural to take $\rho = y_1(x)$. Therefore $\rho \gamma = \frac{y_1}{p \, W}$ is an integrating factor for (1) because it adopts the structure (11) of simple resolution if only we know one solution of (2).

From (11):

$$\alpha = \frac{y_1}{W} \, , \qquad \beta = - \frac{y_1'}{W} \, , \qquad \frac{d}{dx} \left[\frac{{y_1}^2}{W} \, \frac{d}{dx} \left(\frac{y}{y_1} \right) \right] = \frac{y_1 \phi}{p \, W} \, , \tag{13}$$

whose integration gives the general solution of (1):

$$y(x) = a_1\, y_1 + a_2\, y_1 \int^x \frac{W}{{y_1}^2}\, d\xi + y_p\,, \tag{14}$$

such that:

$$y_p = y_1 \int^x \frac{W(\eta)}{{y_1}^2(\eta)}\, d\eta \cdot \int^\eta \frac{y_1(\xi)\phi(\xi)}{p(\xi)\, W(\xi)}\, d\xi = y_1 \left[\frac{y_2}{y_1} \int^x \frac{y_1 \phi}{p\, W}\, d\xi - \int^x \frac{y_2 \phi}{p\, W}\, d\xi \right] = (5),$$

where we employed integration by parts; thus (14) is in total harmony with (3) and (5). Therefore, the concepts of exact and self-adjoint operators imply (13) and its resolution gives the particular solution of the variation of parameters method, but our approach no need the ansatz (8).

In our process first we find $\gamma(x)$ such that $\gamma \hat{L}$ is a self-adjoint operator, and after it is multiplied by $\rho(x)$ to obtain an exact operator. Now we return to (1) multiplying by $\lambda(x)$ to convert it to an exact form:

$$\frac{d}{dx}\,(\mu\, y' + \tau\, y) = \lambda\, \phi\,, \tag{15}$$

which is possible if $\lambda(x)$ verifies the differential condition [11]:

$$p\, \lambda'' + (2\, p' - q)\, \lambda' + (p'' - q' + r)\, \lambda = 0\,, \quad \mu = \lambda\, p\,, \quad \tau = \lambda\, q - \mu'\,. \tag{16}$$

In several situations we may have the property [26]:

$$p'' - q' + r = 0\,, \tag{17}$$

then the differential equation (16) admits the solution $\lambda = 1$, therefore $\mu = p$ & $\tau = q - p'$, and (15) adopts the structure:

$$\frac{d}{dx}\left[p^2\, W\, \frac{d}{dx}\left(\frac{y}{p\, W}\right)\right] = \phi\,, \tag{18}$$

whose integration gives the complete solution of (1); in (18) we note the absence of $y_1(x)$. In resumé, if p, q, r verify (17) then to employ (18); to use (13) if (17) is violated, and in both cases to utilize (4).

3. ERMAKOV - MILNE - PINNEY EQUATION

Here we shall apply the analysis developed in Sec. 2 to give a simple deduction of the Lewis constant [15-19] associated to Ermakov-Milne-Pinney equation [20-22]:

$$y'' + r\, y = a\, y^{-3}, \qquad a > 0, \tag{19}$$

which corresponds to (1) with $p = 1$, $q = 0$, $W = 1$, $\phi = a\, y^{-3}$, then from (5):

$$y_p = y_2 \int^x \frac{a y_1}{{y_p}^3}\, d\eta - y_1 \int^x \frac{a y_2}{{y_p}^3}\, d\eta\,, \qquad y_j'' + r\, y_j = 0, \quad j = 1, 2. \tag{20}$$

It is easy to calculate the wronskian:

$$y_p\, y_2' - y_2\, y_p' = -\int^x \frac{a y_2}{{y_p}^3}\, d\eta\,,$$

because $W = 1$, thus:

$$\frac{y_2}{y_p}\frac{d}{dx}\left(\frac{y_2}{y_p}\right) = -\frac{y_2}{{y_p}^3}\int^x a y_2 {y_p}^{-3}\, d\eta = -\frac{1}{2a}\frac{d}{dx}\left(\int^x a y_2 {y_p}^{-3}\, d\eta\right)^2,$$

$$\therefore \quad \frac{d}{dx}\left(\frac{y_2}{y_p}\right)^2 = -\frac{1}{a}\frac{d}{dx}\left(y_p\, y_2' - y_2\, y_p'\right)^2,$$

that is, $\frac{dI}{dx} = 0$ where I is the Lewis invariant [15-19]:

$$I = a\left(\frac{y_2}{y_p}\right)^2 + \left(y_p\, y_2' - y_2\, y_p'\right)^2 \qquad \textit{Constant of motion.} \tag{21}$$

In [27, 28] we find that:

$$y_1 = \frac{y_p}{a^{1/4}} \sin B(x)\,, \qquad y_2 = \frac{y_p}{a^{1/4}} \cos B(x)\,, \qquad B(x) = -\sqrt{a}\int^x \frac{1}{{y_p}^2}\, d\eta + \varphi\,, \tag{22}$$

which verify (20), besides [22]:

$$y_p = \sqrt{b_1\, {y_1}^2 + 2 b_2\, y_1\, y_2 + b_3\, {y_2}^2}\,, \qquad b_1 = b_3 = \sqrt{a}\,, \qquad b_2 = 0\,. \tag{23}$$

In (21) we can employ (22) for y_2 and thus to deduce the exact value of the Lewis constant, $I = \sqrt{a}$.

4. LINEAR DIFFERENTIAL EQUATIONS OF THIRD AND FOURTH ORDER

Now we consider the linear differential equation [23]:

$$u(x)\, y''' + p(x)\, y'' + q(x)\, y' + r(x)\, y = \phi(x), \tag{24}$$

with $y_1(x)$ and $y_2(x)$ verifying the corresponding homogeneous equation:

$$u\, y''' + p\, y'' + q\, y' + r\, y = 0, \tag{25}$$

and we must find one more solution of (25) and the particular solution of (24). In fact, it is possible to prove that:

$$y_3(x) = y_2(x)\int^x \frac{\omega\, y_1}{(\omega_{12})^2}\, d\eta - y_1(x)\int^x \frac{\omega\, y_2}{(\omega_{12})^2}\, d\eta, \qquad \omega = \exp\left(-\int^x \frac{p}{u}\, d\xi\right), \tag{26}$$

$$y_p(x) = y_1(x)\int^x \frac{\omega_{23}\,\phi}{\omega\, u}\, d\eta + y_2(x)\int^x \frac{\omega_{31}\,\phi}{\omega\, u}\, d\eta + y_3(x)\int^x \frac{\omega_{12}\,\phi}{\omega\, u}\, d\eta, \tag{27}$$

with the wronskians:

$$\omega_{ij} = -\omega_{ji} = y_i\, y_j' - y_j\, y_i', \qquad i \neq j. \tag{28}$$

It is important to note the identities:

$$y_1\,\omega_{23} + y_2\,\omega_{31} + y_3\,\omega_{12} = 0, \quad y_1'\,\omega_{23} + y_2'\,\omega_{31} + y_3'\,\omega_{12} = 0,$$
$$y_1''\,\omega_{23} + y_2''\,\omega_{31} + y_3''\,\omega_{12} = \omega. \tag{29}$$

For the case of the linear differential equation of fourth order [24]:

$$v(x)\, y'''' + u(x)\, y''' + p(x)\, y'' + q(x)\, y' + r(x)\, y = \phi(x), \tag{30}$$

we know the solutions $y_j(x)$, $j = 1, 2, 3$ of the corresponding homogeneous equation (HE), then:

$$y_4(x) = y_1(x)\int^x \frac{\omega_{23}\,\tilde{\omega}}{(\omega_{123})^2}\, d\eta + y_2(x)\int^x \frac{\omega_{31}\,\tilde{\omega}}{(\omega_{123})^2}\, d\eta + y_3(x)\int^x \frac{\omega_{12}\,\tilde{\omega}}{(\omega_{123})^2}\, d\eta, \tag{31}$$

$$y_p(x) = -y_1(x)\int^x \frac{\omega_{234}\,\phi}{\tilde{\omega}\, v}\, d\eta + y_2(x)\int^x \frac{\omega_{341}\,\phi}{\tilde{\omega}\, v}\, d\eta - y_3(x)\int^x \frac{\omega_{412}\,\phi}{\tilde{\omega}\, v}\, d\eta + y_4(x)\int^x \frac{\omega_{123}\,\phi}{\tilde{\omega}\, v}\, d\eta, \tag{32}$$

verify the associated HE and (30), respectively, such that:

$$\tilde{\omega} = \exp\left(-\int^x \frac{u}{v}\, d\xi\right), \qquad \omega_{ijk} = \begin{vmatrix} y_i & y_j & y_k \\ y_i' & y_j' & y_k' \\ y_i'' & y_j'' & y_k'' \end{vmatrix}, \tag{33}$$

with the interesting identities:

$$-y_1\,\omega_{234} + y_2\,\omega_{341} - y_3\,\omega_{412} + y_4\,\omega_{123} = 0,$$
$$-y_1'\,\omega_{234} + y_2'\,\omega_{341} - y_3'\,\omega_{412} + y_4'\,\omega_{123} = 0,$$
$$-y_1''\,\omega_{234} + y_2''\,\omega_{341} - y_3''\,\omega_{412} + y_4''\,\omega_{123} = 0,$$
$$-y_1'''\,\omega_{234} + y_2'''\,\omega_{341} - y_2'''\,\omega_{412} + y_4'''\,\omega_{123} = \tilde{\omega}. \tag{34}$$

5. CONCLUSIONS

Our approach based in exact and self-adjoint operators justifies the ansatz utilized in the variation of parameters method. Besides, our process has strong support in the concept of wronskian, which generates applications to important laws as the Ermakov-Milne-Pinney equation. We consider that is easy to generalize our technique to linear differential equations of third and fourth order. The self-adjoint operators are important in quantum mechanics and in the construction of Green's functions [11, 12], then there our process is interesting to solve the corresponding differential equations.

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