Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 105 (2018) 197-203 EISSN 2392-2192

SHORT COMMUNICATION

Continuity equations in curved spaces

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ABSTRACT

In any Riemannian 4-space, we deduce continuity equations which could be interpreted as conservation laws for the energy and momentum of the gravitational field, with special emphasis in general relativity.

***Keywords*:** Noether's theorem, Lagrangians in Riemannian spaces, Energy and momentum in curved spaces, Rund-Lovelock's relations

1. INTRODUCTION

For Lagrangians-based theories we exploit from very beginning the transformation properties of fields. In this work, we consider gravitational Lagrangians:

$$L = L(g_{ab}; g_{ab,c}; g_{ab,cd}) , \tag{1}$$

(Received 10 July 2018; Accepted 21 July 2018; Date of Publication 25 July 2018)

where: g_{ij} is the metric tensor and $,a = \partial / \partial x^a$; L is a scalar density of weight one [1] under an arbitrary coordinate transformation $\bar{x}^i = \bar{x}^i(x^j)$:

$$L = J\left(\frac{\bar{x}}{x}\right)\bar{L} \quad . \tag{2}$$

Here we shall employ the property (2) to obtain (for general relativity) in natural manner the energy-momentum pseudotensors of Einstein [2-8], Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19], and also the continuity equations of Komar [20], Trautman [21], Du Plessis [22] and Moss [23].

In Sec. 2, we indicate the notation and quantities to employ through the paper. The Sec. 3 is dedicated to the analysis of conservation laws originated from (1), (2) and the Hilbert's variational principle [1, 11, 24, 25]:

$$\delta \int_{V_4} L\sqrt{-g}\, d^4x = 0 \quad . \tag{3}$$

We also mention applications of L in the case of general relativity [4-6, 10, 26].

2. RUND-LOVELOCK IDENTITIES

Rund-Lovelock [1, 24, 25], in their study of variational principles with Lagrangians verifying (1) and (2), showed the importance of the derivatives of L with respect to its arguments:

$$A^{ij} = A^{ji} \equiv \frac{\partial L}{\partial g_{ij}}, \qquad A^{ij,h} = A^{ji,h} \equiv \frac{\partial L}{\partial g_{ij,h}}, \qquad A^{ij,hk} = A^{ji,hk} = A^{ij,kh} \equiv \frac{\partial L}{\partial g_{ij,hk}}, \tag{4}$$

and, in general, only $A^{ij,hk}$ has tensorial character. These quantities have the following properties:

$$A^{ij} = \frac{L}{2} g^{ij} + \frac{4}{3} A^{hk,jr} R_{k\ \mathrm{hr}}^{\ \mathrm{i}} - \Gamma^i{}_{rn}\Gamma^r{}_{km} A^{km,rh}, \qquad A^{ij,h} = \Gamma^i{}_{rk} A^{rk,jh} + \Gamma^j{}_{rk} A^{rk,ih} - \Gamma^h{}_{rk} A^{rk,ij},$$

$$A^{ij,hk} = A^{hk,ij}, \qquad A^{ij,hk} + A^{ih,kj} + A^{ik,jh} = 0, \qquad A_r{}^{\mathrm{i,jh}}{}_{,\mathrm{hji}} = 0, \tag{5}$$

with the convention of Dedekind (1868) [27, 28] - Einstein of sum over repeated indices.

On the other hand, the Euler-Lagrange expressions defined by:

$$L^{ij} = A^{ij} - A^{ij,h}{}_{,h} + A^{ij,hk}{}_{,hk} \ , \tag{6}$$

can be written using (5), in the form:

$$L^{ij} = \frac{L}{2} g^{ij} + \frac{2}{3} A^{hk,jr} R_{\mathrm{k}\ \mathrm{hr}}^{\ \mathrm{i}} + A^{ij,hk}{}_{;hk} \quad , \tag{7}$$

where: c represents the covariant derivative. Besides, we have the contracted Bianchi identities:

$$L^{ij}{}_{;j} = 0 \ . \tag{8}$$

As an example, if L is the scalar density of weight one corresponding to general relativity [4-6, 10, 26]:

$$L = \sqrt{-g} R, \qquad \mathrm{g} = \det(\mathrm{g_{ab}}) \quad , \tag{9}$$

where: $R \equiv R^{ij}{}_{ji}$ is the scalar curvature, then:

$$A^{ij,km} = \frac{1}{2}\sqrt{-g}(2g^{ij}g^{km} - g^{im}g^{jk} - g^{ik}g^{jm}) \ . \tag{10}$$

Thus, (7) and (10) imply that the Euler-Lagrange relations are proportional to the Einstein tensor:

$$L^{ij} = -\sqrt{-g} G^{ij} \ , \tag{11}$$

which satisfies (8) because $G^{ab}{}_{;b} = 0$.

3. CONTINUITY EQUATIONS IN RIEMANNIAN SPACES

The variational principle (3) is invariant under general transformations $\bar{x}^i = \bar{x}^i(x^j)$, in particular, we can use infinitesimal coordinate changes:

$$\bar{x}^i = x^i + \varepsilon^i \xi^i \left(x^i\right) \ , \tag{12}$$

without sum over i, and ε^i denoting small constant parameters. The Noether's theorem [29-38] establishes that each continuous symmetry transformation which leaves the corresponding variational principle invariant, implies a conservation law, and hence, a constant of motion. Here we employ the Noether's theorem via the approach of Lanczos [39-42]: we apply the eq. (12) to eq. (3) but now considering that ε^i are new variational variables, then the Lagrange equations for ε^i give the continuity equations of Noether. Thus, it is possible to deduce the following important relations not found explicitly in the literature on gravitational energy-momentum pseudotensors [43-46]:

$$\left(B_r{}^i \xi^r - U_r{}^{ij} \xi^r{}_{,j} - \frac{1}{2} A_r{}^{i,jh} \xi^r{}_{,jh} \right)_{,i} = 0 \quad , \tag{13}$$

where:

$$U_r{}^{ij} = A_r{}^{i,jh}{}_{;h} + \Gamma_{krh} A^{kj,ih} - \frac{1}{2} \Gamma^i{}_{hk} A_r{}^{j,hk} \, , \tag{14}$$

$$B_r{}^i = U_r{}^{ij}{}_{,j} = -\frac{L}{2}\delta^i_r + L_r{}^i + \frac{1}{2}(A^{jh,i} - A^{jh,ki}{}_{,k}) g_{jh,r} + \frac{1}{2} A^{hk,ij} g_{kh,jr} \quad . \tag{15}$$

Besides, with (8) and (15), it is easy to obtain the conservation law:

$$B_r{}^i{}_{,i} \equiv L_r{}^i{}_{;i} = 0 \ . \tag{16}$$

In the case of general relativity theory, L given by (9), and a complete study of (13) when ξ^r is a vectorial field, leads to results of Komar [20] and Du Plessis [22], and if ξ^r is a Killing vector, then it is also possible to deduce the relations of Trautman [21] and Moss [23]. On the other hand, (16) allows to construct the energy-momentum pseudotensors of Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19].

Sometimes in the Einstein theory, we use the Lagrangian [4, 6, 10, 11]:

$$\bar{L} = \sqrt{-g}\, g^{ab} (\Gamma^i{}_{ab} \Gamma^j{}_{ij} - \Gamma^i{}_{ja} \Gamma^j{}_{ib}), \tag{17}$$

such that $\sqrt{-g}R = \bar{L} + (ordinary\ divergence)$, then the empty field equations are the same for (9) and (17), besides $\partial \bar{L} / \partial g_{ij,hk} = 0$. However, $\bar{L}$ satisfies (2) when ξ^r are constants, therefore (13) is equivalent to $B_r{}^i{}_{,i} = 0$ and from (15):

$$B_r{}^i = 8\pi \sqrt{-g}\, t_r{}^i = \frac{1}{2} \left(\frac{\partial \bar{L}}{\partial g_{jh,i}} g_{jh,r} - \bar{L} \delta^i_r \right), \tag{18}$$

where: $t_r{}^i$ is the canonical energy-momentum pseudotensor of Einstein [2-8, 47]. Thus, the conservation law $t_r{}^i{}_{,i} = 0$ is implied by the translational invariance of $\bar{L}$.

4. CONCLUSIONS.

Therefore, the Lanczos technique [39-42] for the Noether theorem gives the continuity equations (13) and (16) which have total information on energy-momentum quantities for gravitational theories in Riemannian spaces. Thus, the energy-momentum can be regarded as the most fundamental conserved quantity being associated with a symmetry of the spacetime geometry.

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