SHORT COMMUNICATION

Continuity equations in curved spaces

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ABSTRACT

In any Riemannian 4-space, we deduce continuity equations which could be interpreted as conservation laws for the energy and momentum of the gravitational field, with special emphasis in general relativity.

Keywords: Noether’s theorem, Lagrangians in Riemannian spaces, Energy and momentum in curved spaces, Rund-Lovelock’s relations

1. INTRODUCTION

For Lagrangians-based theories we exploit from very beginning the transformation properties of fields. In this work, we consider gravitational Lagrangians:

\[ L = L(g_{ab}, g_{ab,c}, g_{ab,cd}) , \]  

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where: \( g_{ij} \) is the metric tensor and \( a = \partial f/\partial x^a \); \( L \) is a scalar density of weight one [1] under an arbitrary coordinate transformation \( \bar{x}^i = \bar{x}^i(x^i) \):

\[
L = J \left( \frac{\bar{x}}{x} \right) \bar{L}.
\] (2)

Here we shall employ the property (2) to obtain (for general relativity) in natural manner the energy-momentum pseudotensors of Einstein [2-8], Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19], and also the continuity equations of Komar [20], Trautman [21], Du Plessis [22] and Moss [23].

In Sec. 2, we indicate the notation and quantities to employ through the paper. The Sec. 3 is dedicated to the analysis of conservation laws originated from (1), (2) and the Hilbert’s variational principle [1, 11, 24, 25]:

\[
\delta \int_V L \sqrt{-g} d^4x = 0. \tag{3}
\]

We also mention applications of \( L \) in the case of general relativity [4-6, 10, 26].

2. RUND-LOVELOCK IDENTITIES

Rund-Lovelock [1, 24, 25], in their study of variational principles with Lagrangians verifying (1) and (2), showed the importance of the derivatives of \( L \) with respect to its arguments:

\[
A^i = A^i = \frac{\partial L}{\partial g_{ij}}, \quad A^{ij,h} = A^{ij} = \frac{\partial L}{\partial g_{ij,h}}, \quad A^{ij,hk} = A^{ij} = \frac{\partial L}{\partial g_{ij,hk}},
\] (4)

and, in general, only \( A^{ij,hk} \) has tensorial character. These quantities have the following properties:

\[
A^i = \frac{L}{2} g^{ij} + \frac{4}{3} A^{hk,jr} R^i_{hr} - \Gamma^i_{rn} \Gamma^{rn}_{km} A^km, rh, \quad A^{ij,h} = \Gamma^i_{rkj} A^r, jh + \Gamma^j_{rk} A^rk, jh - \Gamma^h_{rk} A^rk, ij,
\]

\[
A^{ij, hh} = A^{hh, ij}, \quad A^{ij, hk} + A^{ih, kj} + A^{ik, jh} = 0, \quad A_r^{i, jh, hj} = 0,
\] (5)

with the convention of Dedekind (1868) [27, 28] - Einstein of sum over repeated indices.

On the other hand, the Euler-Lagrange expressions defined by:

\[
L^{ij} = A^{ij} - A^{ij, h} + A^{ij, hk},
\] (6)
can be written using (5), in the form:

\[ L^j = \frac{L}{2} g^{ij} + \frac{2}{3} A^{hk,jr} R_{khr}^i + A^{ij,hk}, \]  

(7)

where: \( c \) represents the covariant derivative. Besides, we have the contracted Bianchi identities:

\[ L^j.j = 0. \]  

(8)

As an example, if \( L \) is the scalar density of weight one corresponding to general relativity [4-6, 10, 26]:

\[ L = \sqrt{-g} R, \quad g = \det(g_{ab}), \]  

(9)

where: \( R = R^j.j \) is the scalar curvature, then:

\[ A^{ij,km} = \frac{1}{2} \sqrt{-g} \left( 2g^{ij} g^{km} - g^{im} g^{jk} - g^{ik} g^{jm} \right). \]  

(10)

Thus, (7) and (10) imply that the Euler-Lagrange relations are proportional to the Einstein tensor:

\[ L^j = -\sqrt{-g} G^{ij}, \]  

(11)

which satisfies (8) because \( G^{ab,\,b} = 0 \).

3. CONTINUITY EQUATIONS IN RIEMANNIAN SPACES

The variational principle (3) is invariant under general transformations \( \bar{x}^i = \bar{x}^i(x^j) \), in particular, we can use infinitesimal coordinate changes:

\[ \bar{x}^i = x^i + \varepsilon^i (x^j), \]  

(12)

without sum over \( i \), and \( \varepsilon^i \) denoting small constant parameters. The Noether’s theorem [29-38] establishes that each continuous symmetry transformation which leaves the corresponding variational principle invariant, implies a conservation law, and hence, a constant of motion. Here we employ the Noether’s theorem via the approach of Lanczos [39-42]: we apply the eq. (12) to eq. (3) but now considering that \( \varepsilon^i \) are new variational variables, then the Lagrange equations for \( \varepsilon^i \) give the continuity equations of Noether. Thus, it is possible to deduce the following important relations not found explicitly in the literature on gravitational energy-momentum pseudotensors [43-46]:

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\[
\left( B_{r}^{i} \xi^{r} - U_{r}^{i} \xi^{r} j - \frac{1}{2} A_{r}^{i,jh} \xi^{h} \right) = 0 ,
\]
where:
\[
U_{r}^{i,j} = A_{r}^{i,jh} + \Gamma_{kh}^{i,jh} A_{r}^{k,i} - \frac{1}{2} \Gamma_{hk}^{i,jh} A_{r}^{i,jk} ,
\]
\[
B_{r}^{i} = U_{r}^{i,j} = -\frac{L}{2} \delta^{i} + L_{r}^{i} + \frac{1}{2} ( A^{j,k} - A^{j,k} ) g_{j,k} + \frac{1}{2} A^{h,k} g_{h,j} .
\]

Besides, with (8) and (15), it is easy to obtain the conservation law:
\[
B_{r}^{i} \equiv L_{r}^{i} = 0 .
\]

In the case of general relativity theory, \( L \) given by (9), and a complete study of (13) when \( \xi^{r} \) is a vectorial field, leads to results of Komar [20] and Du Plessis [22], and if \( \xi^{r} \) is a Killing vector, then it is also possible to deduce the relations of Trautman [21] and Moss [23]. On the other hand, (16) allows to construct the energy-momentum pseudotensors of Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19].

Sometimes in the Einstein theory, we use the Lagrangian [4, 6, 10, 11]:
\[
\tilde{L} = \sqrt{-g} g^{ab} ( \Gamma_{ab}^{j} - \Gamma_{aj}^{j} ) ,
\]
such that \( \sqrt{-g} R = \tilde{L} + (\text{ordinary divergence}) \), then the empty field equations are the same for (9) and (17), besides \( \partial \tilde{L} / \partial g_{j,kh} = 0 \). However, \( \tilde{L} \) satisfies (2) when \( \xi^{r} \) are constants, therefore (13) is equivalent to \( B_{r}^{i} = 0 \) and from (15):
\[
B_{r}^{i} = 8\pi \sqrt{-g} t_{r}^{i} = \frac{1}{2} \left( \frac{\partial \tilde{L}}{\partial g_{j,k}} g_{j,k} - \tilde{L} \delta^{i} \right) ,
\]
where: \( t_{r}^{i} \) is the canonical energy-momentum pseudotensor of Einstein [2-8, 47]. Thus, the conservation law \( t_{r}^{i} = 0 \) is implied by the translational invariance of \( \tilde{L} \).

4. CONCLUSIONS.

Therefore, the Lanczos technique [39-42] for the Noether theorem gives the continuity equations (13) and (16) which have total information on energy-momentum quantities for gravitational theories in Riemannian spaces. Thus, the energy-momentum can be regarded as
the most fundamental conserved quantity being associated with a symmetry of the spacetime geometry.

References


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