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SHORT COMMUNICATION

Propagators in quantum mechanics

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ABSTRACT

We consider one-dimensional non-relativistic quantum mechanics to exhibit that the propagators (Green's functions) for free particle, linear potential and harmonic oscillator, are obtainable from purely classical means.

***Keywords*:** Propagators in quantum mechanics, Green's functions

1. INTRODUCTION

The wave function $\psi(x,t)$ satisfies the Schrödinger equation [1]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi\,, \tag{1}$$

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which accepts solution via Green's technique [2]:

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,x',t)\, \psi(x',0)\, dx', \tag{2}$$

where: $K(x,x',t)$ is the corresponding propagator for the potential $V(x)$, with the properties:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - i\hbar\frac{\partial}{\partial t}\right) K(x,x',t) = 0, \quad lim_{\,t\to 0}\, K(x,x',t) = \delta(x-x'). \tag{3}$$

In according with Dirac [3, 4] and Feynman [5-7]:

$$K(x,x',t) \;\propto\; \exp\left(\frac{i}{\hbar}\int_0^t L\, dt\right), \tag{4}$$

being L the Lagrangian of the system under study. In (4) are important all paths with initial and final points $(x',0)$ and (x,t), respectively; in this connection we remember the comment from Sommerfeld [8]: 'The causality of the 20th century must not limit itself to the initial state, but must take end-state into consideration as an equally decisive moment'.

Here we determine the classical action integral $\int_0^t L\, dt$ associated with the path actually traversed between the points $(x',0)$ and (x,t), and after we eliminate the constants of integration to establish a posteriori the democracy of all paths between the given events. This approach affords a purely classical means of explicit evaluation of the propagator. In Sec. 2 we exhibit this elementary method to construct the propagators for free particle, harmonic oscillator and constant external field, where $V(x)$ is, at most, a second degree polynomial in x.

2. ELEMENTARY PROCEDURE TO CALCULATE THE PROPAGATOR

Here we find the propagator for some simple systems:

a). $V(x) = 0$.

For free particle we have that $\exp\left(\frac{i}{\hbar}\int_0^t \frac{p^2}{2m} dt\right) = \exp\left(\frac{i}{\hbar}\frac{p^2}{2m} t\right)$ because the linear momentum $p = m\frac{x-x'}{t}$ is a constant, hence from (4):

$$K(x,x',t) = N \exp\left[\frac{im}{2\hbar t}(x-x')^2\right], \tag{5}$$

which satisfies the Schrödinger equation. The relation:

$$\delta(x-a) = lim_{\,\sigma\to 0}\, \frac{1}{\sigma\sqrt{\pi}} \exp\left[-\frac{(x-a)^2}{\sigma^2}\right], \tag{6}$$

permits to obtain N in (5) to verify (3), therefore the propagator for free particle is given by [1, 9-16]:

$$K(x,x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[-\frac{m}{2i\hbar t}(x-x')^2\right]. \tag{7}$$

b). $V(x) = kx$.

For this linear potential the classical equations of motion give $\dot{x} = c - \frac{k}{m}t,\ x = x' - \frac{k}{2m}t^2 + ct,\ c =$ constant, $L = \frac{m}{2}\dot{x}^2 - kx$, then from (4):

$$K(x.x',t) = N\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2t}(x-x')^2 - \frac{k}{2}(x+x')\,t - \frac{k^2}{24m}t^3\right]\right\}, \tag{8}$$

which verifies the Schrödinger equation. From (6) and (8) we determine the value $N = \sqrt{\frac{m}{2\pi i\hbar t}}$ to satisfy (3), thus we reproduce the expression reported in the literature [9, 10, 15, 17] for the propagator under a constant external field; the relation (8) with $k = 0$ gives (7) for free particle.

c). $V(x) = \frac{1}{2}m\omega^2x^2$.

For the harmonic oscillator the classical solution is given by:

$$x(t) = A\cos(\omega t+\varphi) = x'\cos(\omega t) - A\sin\varphi\cdot\sin(\omega t), \quad x' = A\cos\varphi, \tag{9}$$

then

$$\int_0^t L\,dt = \int_0^t\left(\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2x^2\right)dt = -\frac{m\omega}{2}\left[xx'\sin(\omega t) + A\sin\varphi\cdot(x\cos(\omega t) - x')\right],$$

hence (4) implies:

$$K(x,x',t) = N\exp\left\{\frac{im\omega}{2\hbar\sin(\omega t)}\left[\left(x^2+x'^2\right)\cos(\omega t) - 2xx'\right]\right\}, \tag{10}$$

because from (9) we have that $A\sin\varphi = \frac{[x'\cos(\omega t) - x]}{\sin(\omega t)}$. This expression (10) satisfies the Schrödinger equation and it is easy to see that $N = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}$ to verify (3) in according with (6), thus we have obtained the formula for the harmonic oscillator propagator [1, 9-12, 14-16, 18-23].

3. CONCLUSIONS

The Lagrangians here considered are quadratic in the velocities where $V(x)$ is, at most, a second degree polynomial [10, 12, 24, 25]. It is simple to check that the propagators (7), (8) and (10) satisfy the Trotter formula [12, 26, 27]:

$$K(x,x',t) = \left(\frac{m}{2\pi i\hbar t}\right)^{r/2} \exp\left\{\frac{i}{2\hbar}\left[\frac{m}{t}(x-x')^2 - \big(V(x)+V(x')\big)t\right]\right\}, \tag{11}$$

for $t \ll 1$ with $r = 1$. We find that in the three cases here studied the Fujiwara's result [9] is verified:

$$N = \sqrt{\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x \partial x'}\int_0^t L\, dt}\ . \tag{12}$$

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