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SHORT COMMUNICATION

Composite Labelling of Graphs - II

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ABSTRACT

Composite labelling is a bijective function from the set of all vertices and edges to the set of all natural numbers up to the sum of the number of all vertices and edges in a graph. In this paper, the composite labelling of the Cartesian and Tensor Products of paths, Corona of ismorphic cycles and generalised Petersen graph are explored.

***Keywords*:** Labelling of Graphs, Total Labelling, Composite Labelling, GCD

1. INTRODUCTION

Graph labelling is an assignment of labels to vertices or edges subject to some condition and the graphs then are called labelled graphs. Any discrete structure that can be modeled with vertices and edges has some labels associated with such as their names, weights, heights etc. However, for the purpose of programming, we require numeric labels. There are several graph labellings such as prime labelling [1, 7, 10], prime and coprime labelling [2], divisor labeling [5], sum labelling [9] etc. that draw light into many concepts in number theory. For

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an almost exhaustive survey of graph labelling read [8]. For standard concepts and notations Graph Theory, we use [3, 4, 11].

Ponnore and Kureethara introduced the composite labelling of graphs. Let G (V, E) be an undirected simple connected graph of order n and size m. Let *u*, *v*, *w* ∈ V(G) such that *uv* and *vw* are edges incident with the common vertex *v*. To be more accurate, *uvw* is a path of length two in the graph G. A composite labelling is a bijective function *f*:(V(G)∪E(G))→{1, 2, 3, ..., *m*+*n*} such that gcd (f(*uv*), f(*vw*)) ≠1 [12]. Composite graphs are those graphs that have composite labellings. All graphs are not composite graphs. All cycles are composite whereas all complete graphs with at least four vertices are not composite. In [12], we see some standard results associated with composite labelling.

We state them without proofs.

Theorem A All trees admit composite labelling [12].

Theorem B All cycles admit composite labelling [12].

Theorem C All unicyclic graphs are composite graphs [12].

Theorem D The ladder graph L_n, *i.e.*, $P_n \square P_2$ is composite [12].

Throughout this paper, P_n and C_n denote path and cycle on n vertices, respectively and $P_{n,k}$ denotes the generalized Petersen Graph. Let S be the set of all natural numbers up to the sum of the number of all vertices and edges in a graph. Let S_e and S_o be the even and odd integers in S, respectively. We also define some other subsets of S as: (1) $S_{\alpha e}$ = {x ∈ S_e | x is a multiple of α}, (2) $S_{n\alpha e} = S_e \backslash S_{\alpha e}$, (3) $S_{\alpha o}$ = {x ∈ S_o | x is a multiple of α} and (4) $S_{n\alpha o} = S_o \backslash S_{\alpha o}$.

2. RESULTS

We now find some more graphs that are composite.

2. 1. Grid Graph

The Cartesian product $P_n \square P_m$ where m+n >2 is called a grid graph.

Theorem 2.1. The grid graph $P_n \square P_m$ is composite.

Proof. Let $P_n \square P_m$ be the grid graph ∀ *n*, *m* ∈ ℕ. Then $P_n \square P_m$ has nm vertices and 2*nm*-(*m*+*n*) edges. Let S = {1, 2, . . . , 3*nm*-(*m*+*n*)}. If n or m is equal to 1, then the graph is a path, hence admits composite labelling. So let us consider the case where *n*, *m*>2. We use all the integers in S to label the graph.

If *m* or *n* is odd then S has $\frac{[3nm-(m+n)]-1}{2}$ even integers and $\frac{[3nm-(m+n)]+1}{2}$ odd integers. If both *m* and *n* are even, then S has $\frac{3nm-(m+n)}{2}$ each even and odd integers.

As $nm < \frac{[3nm-(m+n)]-1}{2} < \frac{[3nm-(m+n)]}{2} < \frac{[3nm-(m+n)]+1}{2} < 2nm\text{-}(m+n)$, vertices of $P_n \square P_m$ can be labelled with odd integers always. However, not all edges can be labelled with even integers alone. We require some odd integers as well to complete the composite labelling.

Now, $|S_{n3e}| = \frac{[3nm-(m+n)]-1}{2} - \frac{[3nm-(m+n)]-1}{6}$, if n or m is odd. Similarly, $|S_{n3e}|=\frac{[3nm-(m+n)]-1}{2} - \frac{3nm-(m+n)}{6}$, if n and m are even. Let $|S_{n3e}| = p$ and $|S_{3e}| = q$.

Start labelling from the top of the grid using even integers of S_{n3e} sequentially in the ascending order. After labelling p edges, continue labelling the edges using the integers in the set S_{3e}.

Hence a total of p+q edges are labelled. i.e., $|S_e|$ edges are labelled. Then label the remaining m (total number of edges) - $|S_{3e}|$ edges with the odd integers of S_{3o}. At last, label the vertices of graph using odd integers other than that of labelled ones.

Let us see an illustration. Consider the graph $P_5 \square P_5$. It has 25 vertices and 40 edges. Hence, S = {1, 2, . . . , 65}. The other related sets are S_{n3e} = {2, 4, 8 10, 14, 16, 20, 22, 26, 28, 32, 34, 38, 40, 44, 46, 50, 52, 56, 58, 62, 64}, S_{3e} = {6, 12, 18, 24, 30, 36, 42, 48, 54, 60} and S_{3o}= {3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63}.

The odd integers taken from $S_{3o}$are 3, 9, 15, 21, 27, 33, 39 and 45. All the remaining odd integers are used to label the vertices. The diagram shows the labelling of $P_5 \square P_5$.

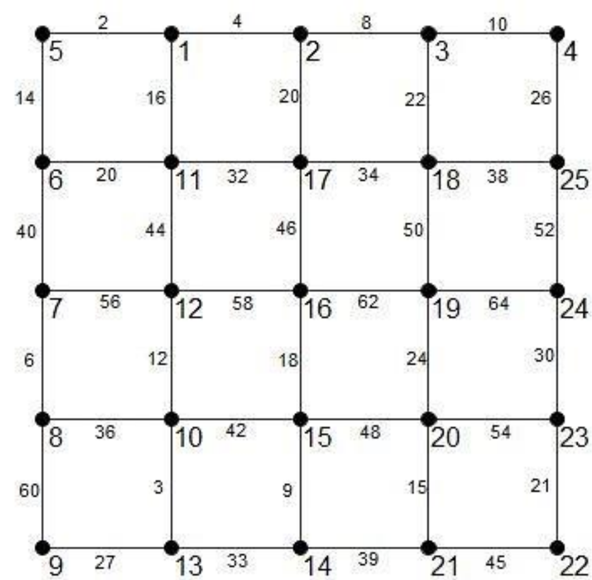

Figure 1. Composite Labelling of the graph $P_5 \square P_5$.

2. 2. Corona of Cycles

We now consider the corona of graphs. The corona $G_1 o G_2$ of two graphs $G_1$and $G_2$is obtained by taking one copy of G_1 (which has $p_1$vertices) and $p_1$copies of G_2, and then joining he i^{th} vertex of $G_1$to every vertex in the i^{th} copy of G_2.We now take G_1 and G_2 as cycles of same length.

Theorem 2.2. The Corona isomorphic cycles is composite.

Proof. Let C_n o C_n denote the corona of cycles of length n. It has n^2 + n vertices and $2n^2$+n edges. Now, S = {1, 2, . . . , $3n^2 + 2n$}.

If n is odd, then S has $\frac{3n^2 + 2n + 1}{2}$ odd integers and $\frac{3n^2 + 2n - 1}{2}$ even integers. If n is even, then S has $\frac{3n^2 + 2n}{2}$ odd and even integers.

As n^2 + n < $\frac{3n^2 + 2n - 1}{2}$ < $\frac{3n^2 + 2n}{2}$ < $\frac{3n^2 + 2n + 1}{2}$ < $2n^2$+n, vertices of C_n o C_n can be labelled with odd integers always but not all the edges can be labelled with even integers alone so we use odd multiples of 3 to complete composite labelling.

If n is odd then we need (total number of edges) m-$|S_e|$ = $\frac{n^2 + 1}{2}$ number of S_{3o}. If n is even then we need (total number of edges) m - $|S_e|$ = $\frac{n^2}{2}$ number of S_{3o}.

Start labelling the edges of the corona graph (*i.e.*, outer cycles) with S_{3o}integers, depending on whether n is odd or even. Then label the adjacent edges of edges labelled with S_{3o} integers with S_{3e} integers such that, we can label the next cycle (outer) only if we complete labelling all the edges of previous cycle (outer).

Then label the remaining edges using remaining even multiples in S. At last, label the vertices of graph with remaining odd multiples in S.

Let us now see the composite labelling of C_3 o C_3 with 12 vertices and 21 edges. Hence, S = {1, 2, . . . , 33}, S_{3o}= {3, 9, 15, 21, 27, 33} and S_{3e} = {6, 12, 18, 24, 30}.

Here n is odd, so we label edges using 5 odd multiples of 3.

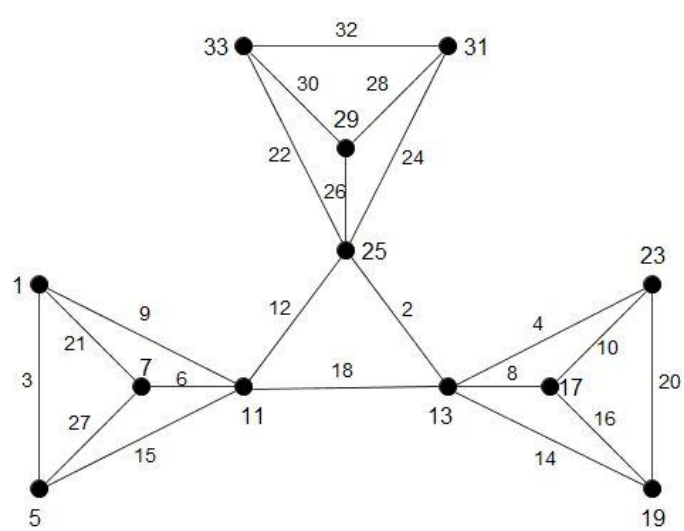

Figure 2. Composite Labelling of the Corona of C_3 and C_3.

2. 3. Generalised Petersen Graph

The generalized Petersen graph $P_{n,k}$is the graph with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1 \ldots, v_n\}$ and edges $\{u_i u_{i+1},\ u_i v_i, v_i v_{i+k}: i = o, \ldots, n-1\}$, where subscripts are to be read modulo n and k < n/2 [6].

Theorem 2.3. Generalized Petersen graph is composite.

Proof. Let $P_{n,k}$ (k = 1 and 2) denote generalized Petersen graph on 2*n* vertices and 3*n* edges. Now, S ={1, 2, . . . , 5*n*}. We use all the integers in S to label $P_{n,k}$.

If *n* is odd then S has $\frac{5n-1}{2}$ even integers and $\frac{5n+1}{2}$ odd integers. If n is even then, S has $\frac{5n}{2}$ even and odd integers. As $2n < \frac{5n-1}{2} < \frac{5n}{2} < \frac{5n+1}{2} < 3n$, vertices of $P_{n,k}$ can be labelled with odd integers always. But, not all edges can be labelled with even integers alone. We require some odd integers as well to complete the composite labelling. We require m - $|S_e|$ number of odd multiple of 3 to label the edges. Let m - $|S_e|$ = p.

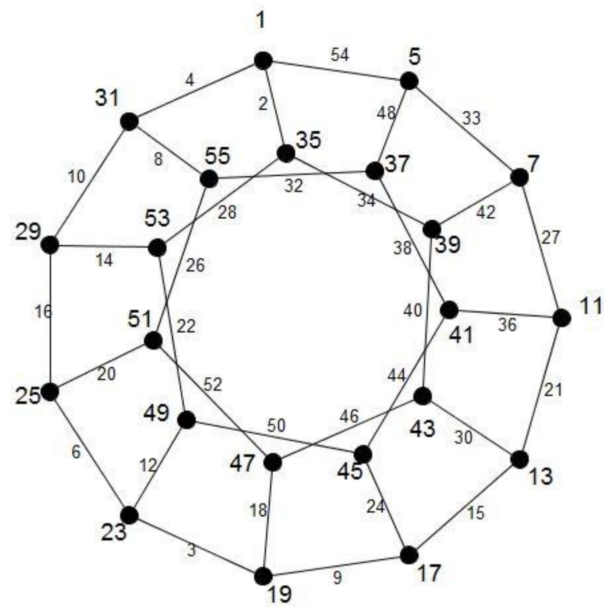

Figure 3. Composite Labelling of $P_{11,2}$

Case 1: 4 < *n* <9.

Start labelling the outer edges with p number of S_{3o}integers sequentially in ascending order such that the P^{th} edge should be labelled with the integer which is a multiple of both 3 and 5. Then label the *p*+3 number of adjacent edges of already labelled outer edges with S_{3e} integers. If *n* = 4, 5 and 7 then only *p*+1 number of edges are labelled S_{3e} integers and if *n*=6, 8 and 9 then only *p*+2 edges are labelled with S_{3e} integers, then label the remaining edges which are adjacent to P^{th} with S_{5e}. therefore We have labelled 2*p*+3 number of edges.

Continue labelling the remaining edges with S_{n3e} integers sequentially in ascending order. At last label the vertices of $P_{n,k}$ with S_o integers other than labelled ones.

Case 2: *n* >10.

Start labelling the outer edges with p number of S_{3o} integers sequentially in ascending order. Continue labelling the p + 3 number of adjacent edges of already labelled outer edges with S_{3e} integers. Therefore, we have labelled 2p + 3 number of edges.

Continue labelling the remaining edges with S_e integers other than labelled ones in sequential ascending order.

At last label the vertices of $P_{n,k}$ with S_o integers other than labelled ones.

Let us now see the composite labelling of $P_{11,2}$ that has 22 vertices and 33 edges. Let S = {1, 2, . . . , 55}, S_{3e} = {6, 12, 18, 24, 30, 36, 42, 48, 54} and S_{3o}= {3, 9, 15, 21, 27, 33, 39, 45, 51}.

The odd integers used are 3, 9, 15, 21, 27 and 33.

2. 4. Tensor Product of Path Graphs

The tensor product $G_1 \wedge G_2$ of connected graph G_1 and G_2 has $V_1 \times V_2$ as vertex set and u = (u_1 , u_2) is adjacent to v = (v_1, v_2) whenever u_1 adjacent v_1 and u_2 adjacent v_2.

We now discuss tensor product (conjunction) of paths and cycles.

Theorem 2.4. Tensor product of path graphs are composite.

Proof. Let $P_n \wedge P_m$ represent the tensor product of P_n and P_m. It has *nm* vertices and 2(*nm* - (*m*+*n*) + 1) edges. Let S = {1, 2, . . . , 3*mn* - 2(*m* + *n*) + 2}. If *n*, *m* <4, then the graph admits composite labelling, since the graph is either unicyclic or the number of edges in the graph is less than number of vertices. So let us consider n >3 and m >5.

If both *m* and *n* are odd, then S is odd and S has $\frac{3mn - 2(m + n) + 3}{2}$ odd integers and $\frac{3mn - 2(m + n)+1}{2}$ even integers. If any one of *m* or *n* is even then S is even and S has $\frac{3mn - 2(m + n) + 2}{2}$ even and odd integers.

As mn < $\frac{3mn - 2(m + n)+1}{2} < \frac{3mn - 2(m + n) + 2}{2} < \frac{3mn - 2(m + n) + 3}{2}$ < 2mn - 2(m + n) + 2, vertices of $P_n \wedge P_m$ can be labelled with odd integers always but not all the edges can be labelled with even integers alone so we use odd multiples of 3 to complete composite labelling. If S has an odd number of integers then we need $\frac{mn - 2(m + n) + 3}{2}$ = P odd multiples of 3. If S has even number of integers then we need $\frac{mn - 2(m + n) + 2}{2}$ = Q odd multiples of 3.
In the tensor product of path graphs, we can notice 2 categories of paths in it. One is facing towards the left of the graph and another one to the right

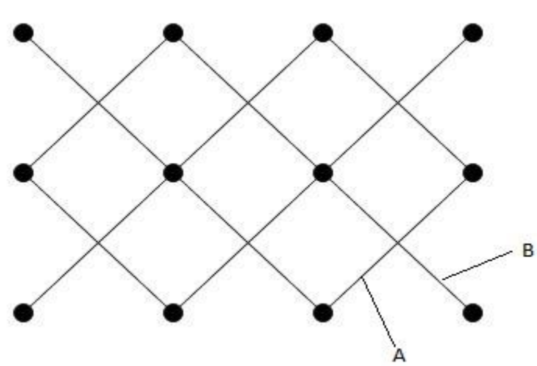

Figure 4. Tensor Product of $P_3 \wedge P_4$

Start labelling from the left side of the graph that is, label the A-paths using P or Q number of S_{3o}that is S_{3o}then label the B-paths and edges in A-path which are adjacent to already labelled A-paths with S_{3e} integers. Continue labelling the remaining edges with remaining S_e integers. At last label the vertices with remaining S_o integers.

Let us now see the composite labelling of $P_4 \wedge P_5$. $P_4 \wedge P_5$ has 20 vertices and 24 edges. Hence S = {1, 2, . . . , 44}, S_{3o}= {3, 9, 15, 21, 27, 33, 39} and S_{3e} = {6, 12, 18, 24, 30, 36, 42}.

Since S has even number of integers, we use 2 odd multiples of 3.

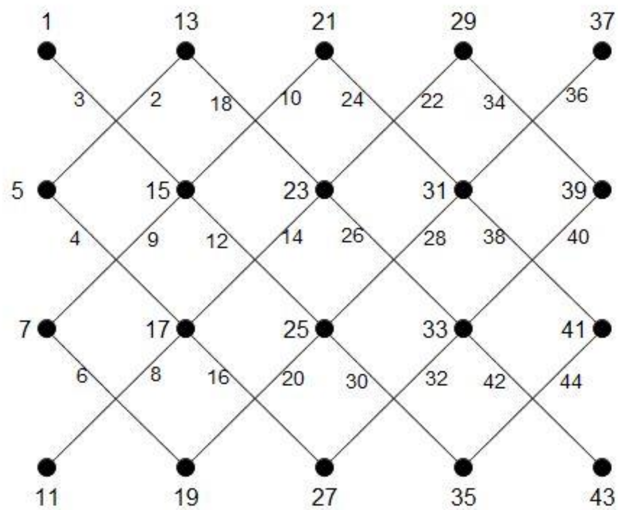

Figure 5. Composite Labelling of $P_4 \wedge P_5$

3. CONCLUSIONS

This is a study of the composite labelling of some well-known graphs. It is found out that the Cartesian Product of Paths, *i.e.*, the grid graph is a composite graph. So also is the Tensor Product of Paths. In both the cases, the maximum degree is 4. The famous 3-regular graph, generalized Petersen graph, is also composite. The corona of two graphs need not be composite in general. Nevertheless, a large family of the corona of graphs is composite. We have shown that corona of two isomorphic cycles is a composite graph. A major turning point in the study of the composite labelling is the critical role played by the number of edges. The more sparse the graph is, the more is its chance to be composite. Hence, a study in that direction will lead to major characterizations in the composite labelling of graphs.

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