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SHORT COMMUNICATION

# Shifted Chebyshev-Lanczos polynomials

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**ABSTRACT**

We show an algorithm for to express $x^n$ in terms of the modified Chebyshev-Lanczos polynomials $T_0^*, T_1^*, ..., T_n^*$, which is important to reduce the degree of a polynomial in the interval [0, 1]. Besides, with the use of $T_2^*$ into logistic map we obtain a simple geometric interpretation of the Feigenbaum´s universal constant.

***Keywords*:** Modified Lanczos polynomials, Feigenbaum constant, Pascal triangle

## 1. INTRODUCTION

In numerical analysis it is necessary to reduce, with small error, the degree of a polynomial in the interval [0, 1], which is possible using the shifted Chebyshev-Lanczos polynomials [1-5] $T_r^*$ defined by:

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$$T_0^* = \frac{1}{2}, \quad T_k^*(x) = T_k(2x-1) = T_{2k}(\sqrt{x}), \quad k = 1,2,... \tag{1}$$

where the polynomials $T_r$ are given by the recurrence relation:

$$T_0 = 1, \ T_1(x) = x, \ T_{k+1}(x) - 2xT_k(x) + T_{k-1}(x) = 0, \tag{2}$$

therefore:

$$T_0^* = \frac{1}{2}, \qquad T_1^* = 2x-1, \qquad T_2^* = 8x^2 - 8x + 1,$$
$$T_3^* = 32x^3 - 48x^2 + 18x - 1, \qquad T_4^* = 128x^4 - 256x^3 + 160x^2 - 32x + 1, \ \ldots \tag{3}$$

During the mentioned reduction process we need the powers of $x$ in terms of $T_r^*$, then from (3):

$$x^0 = 2T_0^*, \ x = \frac{1}{2}(2T_0^* + T_1^*), \ x^2 = \frac{1}{8}(6T_0^* + 4T_1^* + T_2^*),$$
$$x^3 = \frac{1}{32}(20T_0^* + 15T_1^* + 6T_2^* + T_3^*), \quad x^4 = \frac{1}{128}(70T_0^* + 56T_1^* + 28T_2^* + 8T_3^* + T_4^*), \ \ldots \tag{4}$$

in general:

$$\frac{1}{2}(4x)^n = \sum_{k=0}^{n} \frac{(2n)\,!}{k!(2n-k)\,!} T_{n-k}^*(x). \tag{5}$$

In Sec. 2 we construct a Pascal triangle which itself contains the same information than (5). The Sec. 3 shows an algorithm to obtain the expansion of $x^k$ in terms of $T_r^*$ when it is known the corresponding development of $x^{k-1}$. In Sec. 4 we employ $T_2^*$ into quadratic map [6], resulting thus a simple geometrical interpretation of the Feigenbaum constant [7, 8] in terms of the graph for the first iteration.

## 2. PASCAL TRIANGLE FOR $T_k^*$.

In fact, we can write (5) in the form:

| | $T_0^*$ | $T_1^*$ | $T_2^*$ | $T_3^*$ | $T_4^*$ | $\ldots$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}(4x)^0$ | 1 | | | | | |
| $\frac{1}{2}(4x)^1$ | 2 | 1 | | | | |
| $\frac{1}{2}(4x)^2$ | 6 | 4 | 1 | | | |
| $\frac{1}{2}(4x)^3$ | 20 | 15 | 6 | 1 | | |
| $\frac{1}{2}(4x)^4$ | 70 | 56 | 28 | 8 | 1 | |
| $\vdots$ | | | | | | |

(6)

or in function of the column vectors $(\frac{1}{2}(4x)^r)$ and $(T_r^*)$ for a given $n$:

$$(\frac{1}{2}(4x)^k) = \underset{\sim}{A} \cdot (T_k^*) \tag{7}$$

where $\underset{\sim}{A}$ is the $(n+1)x(n+1)$ triangular matrix of coefficients appearing in (6):

$$\underset{\sim}{A} = (a_{jk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & \\ 6 & 4 & 1 & 0 & \\ 20 & 15 & 6 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{8}$$

then $(T_k^*) = \underset{\sim}{A}^{-1} \cdot (\frac{1}{2}(4x)^k)$ reproduces (3). Now the aim is to elaborate a simple and efficient algorithm to construct $\underset{\sim}{A}$, which is equivalent to a Pascal triangle.

## 3. ALGORITHM FOR $\frac{1}{2}(4x)^r$ VS $T_k^*$.

Here we shall show that any row of $\underset{\sim}{A}$ permits to calculate the next row. First we must observe that (5) and (8) imply the relations:

$$a_{jk} = \frac{(2j-2)\,!}{(j-k)\,!\;\;(j+k-2)\,!}, \qquad j,k = 1,...,n+1,$$

(9)

$$a_{jj} = 1, \qquad a_{jk} = 0, \qquad k > j$$

which permits to prove the following expressions not found explicitly in the literature:

$$a_{j1} = 2(a_{j-1\ 1} + a_{j-1\ 2}), \qquad j = 2,...,n+1,$$

(10)

$$a_{jk} = a_{j-1\ k-1} + 2a_{j-1\ k} + a_{j-1\ k+1}, \qquad k > 1.$$

The formulae (10) represent the required algorithm, and their systematic use minimizes the amount of arithmetical computations involved in (5).

## 4. FEIGENBAUM CONSTANT

The logistic map is characterized by the quadratic iteration [6]:

$$\psi(x) = \lambda(x - x^2), \qquad x \in [0,1], \qquad \lambda \in [1,4] \tag{11}$$

The shifted Chebyshev-Lanczos polynomials are useful [5] to reduce the degree of a polynomial in the interval [0, 1], then it is natural to ask what benefit it should report the application of (3) to (11). Therefore, for such goal we put [5] $T_2^* = 0$, then $x^2 = x - \frac{1}{8}$ and thus (11) adopts the form:

$$\psi(x) = \frac{\lambda}{8}. \tag{12}$$

The straight line (12) intersects to the parable (11) in the points:

$$x_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right), \qquad x_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right), \tag{13}$$

from where it is immediate the interesting result:

$$\frac{x_2}{x_1} - \frac{x_2}{x_2 - x_1} = 4.67 \approx \delta = 4.669... \tag{14}$$

that is, the universal constant of Feigenbaum [7, 8] accepts the following geometric interpretation: it is the difference between cocients of intervals associated with the graph of the first iteration in the logistic map [6].

## 5. CONCLUSIONS

We show that it is possible construct a Pascal triangle for the powers of $x$ in terms of the shifted Chebyshev-Lanczos polynomials, which is useful to reduce the degree of a polynomial in the interval [0, 1] with the minimal loss of information [9].

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