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SHORT COMMUNICATION

Differentiation of a Fourier series

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ABSTRACT

It is very known that if the operator $\frac{d}{dx}$ acts on each term into a convergent Fourier series (FS) then it may result a divergent series. This situation is remedied applying the symmetric derivative to FS, which implies the existence of the important Fejér-Lanczos factors. In this note, we show that the orthogonal derivative also leads to these factors.

***Keywords*:** Fourier series, Generalized derivative, Least squares method, Fejér-Lanczos σ-Factors

1. INTRODUCTION

If on the Fourier series [1]:

$$f(x) = \frac{1}{2}\, a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + \mathrm{b_k} \sin(kx)\right] , \tag{1}$$

convergent in $[-\pi, \pi]$, we apply the operator $\frac{d}{dx}$ results:

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$$\frac{d}{dx}\, f(x) = \sum_{k=1}^{\infty} k\, [-\, a_k\, \sin(kx) + \mathrm{b_k} \cos(kx)]\ , \tag{2}$$

which it may be divergent [2, 3]. This problem was remedied by Lanczos [3, 4] with $f'(x)$ defined as a Symmetric Derivative [5, 6]:

$$f'(x) \equiv lim_{\, n\to\infty}\, \frac{1}{\frac{2\pi}{n}} \left[f_n\left(x + \frac{\pi}{n}\right) - f_n\left(x - \frac{\pi}{n}\right)\right]\ , \tag{3}$$

with the partial sums:

$$f_n(x) = g_n(x) + h_n(x)\ ,$$

$$g_n(x) = \frac{1}{2}\, a_0 + \sum_{k=1}^{n} a_k\ \cos(kx)\ , \qquad h_n(x) = \sum_{k=1}^{n} b_k \sin(kx)\ , \tag{4}$$

resulting the convergent expression:

$$f\,'(x) = lim_{\, n\to\infty}\, \sum_{k=1}^{n} \sigma_k \frac{d}{dx}\, [a_k\ \cos(kx) + b_{\mathrm{k}} \sin(kx)]\, , \tag{5}$$

with the Fejér-Lanczos Factors [3, 4, 7-10]:

$$\sigma_0 = 1\, , \qquad \sigma_k = \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}}\ , \qquad k = 1, \ldots, n\, , \qquad \sigma_n = 0\, ; \tag{6}$$

the set of factors σ_k , for a given *n*, is equivalent to a discrete sampling function. This method amounts to a multiplication of the standard Fourier coefficients a_k and b_k by the 'attenuation factors' σ_k .

In (2) and (3) we employ two types of derivatives, however, also there is the orthogonal derivative [6, 11-22] obtained by Lanczos [4] Cioranescu [23] and Haslam-Jones [24], hence it is natural to ask if this ultimate derivative leads to relation (5). The answer is yes, to see the next Section.

2. THE ORTHOGONAL DERIVATIVE

Lanczos [4, 25] used the least squares method of Legendre [26]-Gauss [27]-Laplace [28] to obtain an integral expression for the derivative of a function, that is, differentiation by integration:

$$F\,'(x)\ = lim_{\,\epsilon\to 0}\, \frac{3}{2\,\epsilon^3}\ \int_{-\epsilon}^{\epsilon} t\ F(x+t)\, dt\, , \tag{7}$$

which may be applied to Fourier case:

$$\frac{3}{2\,\epsilon^3}\ \int_{-\epsilon}^{\epsilon} t\, g_n(x+t)\, dt \ \overset{(4)}{=}\ \frac{3}{2\,\epsilon^3}\, \sum_{k=1}^{n} a_k \int_{-\epsilon}^{\epsilon} t\ \cos(kx + kt)\, dt\ ,$$

$$= -3 \sum_{k=1}^{n} a_k \frac{\sin(kx)}{k^2} A_k \quad \text{with} \quad A_k(\epsilon) = \frac{1}{\epsilon^3} [\sin(k\epsilon) - k\,\epsilon \cos(k\epsilon)] \ ; \tag{8}$$

similarly:

$$\frac{3}{2\,\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, h_n(x+t)\, dt \overset{(4)}{=} \frac{3}{2\,\epsilon^3} \sum_{k=1}^{n} b_k \int_{-\epsilon}^{\epsilon} t\ \sin(kx+kt)\, dt \ ,$$

$$= 3 \sum_{k=1}^{n} b_k \frac{\cos(kx)}{k^2} A_k \ . \tag{9}$$

Therefore, the Lanczos derivative applied to partial sum (4) gives, taking $\epsilon = \frac{\pi}{n}$:

$$f\,'(x) = \ lim_{\,n\to\infty} \frac{3}{2\,\epsilon^3} \int_{-\epsilon}^{\epsilon} t\ f_n(x+t)\, dt\,,$$

$$\overset{(8)\text{ and }(9)}{=} \quad lim_{\,n\to\infty}\ 3 \sum_{k=1}^{n} \frac{1}{k^2} A_k \left[-a_k \ \sin(kx) + b_k \cos(kx)\right],$$

$$= lim_{\,n\to\infty} \sum_{k=1}^{n} \frac{3\,A_k}{k^3} \frac{d}{dx} \left[a_k \cos(kx) + b_k \sin(kx)\right] \ , \tag{10}$$

but the Bernoulli–Hôpital rule permits to observe the behavior:

$$A_k\left(\epsilon = \frac{\pi}{n}\right) \xrightarrow[n>>1]{} \frac{k^3}{3} \frac{\sin(k\,\epsilon)}{k\,\epsilon} = \frac{k^3}{3} \frac{\sin\left(\frac{k\,\pi}{n}\right)}{\frac{k\,\pi}{n}} \overset{(6)}{=} \frac{k^3}{3} \sigma_k \ , \tag{11}$$

and this value employed in (10) implies (5), q.e.d.

Thus, it is proved that the symmetric and Cioranescu-(Haslam-Jones)-Lanczos derivatives imply the same expression for the derivative of an infinite Fourier series, with the important participation of the Fejér–Lanczos factors.

3. CONCLUSIONS

The orthogonal derivative is a generalization of the symmetric derivative and this is a generalization of the standard derivative. The orthogonal derivative acts as a smoothing (integrating filter):

That is why the highest frequencies of the input are being suppressed in the deduction in Sec. 2, and so this creates also the suppression of the Gibbs phenomenon [1, 4, 7, 8, 29-32]. The Legendre polynomials [1, 4, 33, 34] can be employed to extend the method of Cioranescu-(Haslam-Jones)-Lanczos to cover orthogonal derivatives of higher orders [17, 35, 36].

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