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SHORT COMMUNICATION

An alternative to Gower's inverse matrix

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ABSTRACT

We show that the Faddeev-Sominsky's process allows construct a natural inverse for any square matrix, which is an alternative to the inverse obtained by Gower.

***Keywords*:** Gower's pseudoinverse matrix, Characteristic equation, Eigenvalue problem, Adjoint matrix, Faddeev-Sominsky's method, Leverrier-Takeno's algorithm

1. INTRODUCTION

For an arbitrary matrix $\mathbf{A}_{nxn} = (\mathrm{A}^i{}_j)$ its characteristic equation [1-3]:

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{1}$$

can be obtained, through several procedures [1, 4-7], directly from the condition det $(\mathrm{A}^i{}_j - \lambda\delta^i_j) = 0.$ The approach of Leverrier-Takeno [4, 8-12] is a simple and interesting technique to construct (1) based in the traces of the powers $\mathbf{A}^r,\ r = 1, \dots, n.$

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On the other hand, it is well known that an arbitrary matrix satisfies (1), which is the Cayley-Hamilton-Frobenius identity [1-3]:

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}. \tag{2}$$

If $\mathbf{A}$ is non-singular (that is, det $\mathbf{A} \neq 0$), then from (2) we obtain its inverse matrix:

$$\mathbf{A}^{-1} = -\frac{1}{a_n}(\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \cdots + a_{n-1}\mathbf{I}), \tag{3}$$

where: $a_n \neq 0$ because $a_n = (-1)^n \det \mathbf{A}$.

Faddeev-Sominsky [13-15] proposed an algorithm to determine $\mathbf{A}^{-1}$ in terms of $\mathbf{A}^r$ and their traces, which is equivalent [16] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix $\mathbf{A}$, to see Sec. 2. The condition $\det \mathbf{A} \neq 0$ means $p \equiv rank\ \mathbf{A} = n$, then it is immediate the quest of an inverse of $\mathbf{A}$ when $1 \leq p \leq (n-1)$, in fact, in Sec. 3 we exhibit a pseudoinverse $\mathbf{A}^-$ with the properties [17, 18]:

$$\mathbf{A}\,\mathbf{A}^-\mathbf{A} = \mathbf{A}, \tag{4}$$

$$\mathbf{A}^-\mathbf{A}\,\mathbf{A}^- = \mathbf{A}^-, \tag{5}$$

which we consider more simple and natural than the Gower's inverse [19].

2. LEVERRIER-TAKENO AND FADDEEV-SOMINSKY TECHNIQUES

If we define the quantities:

$$a_0 = 1, \qquad s_k = tr\ \mathbf{A}^k, \quad k = 1, 2, \ldots, n \tag{6}$$

then the process of Leverrier-Takeno [4, 8-12] implies (1) wherein the a_i are determined with the recurrence relation:

$$r\,a_r + s_1\,a_{r-1} + s_2\,a_{r-2} + \ldots + s_{r-1}\,a_1 + s_r = 0, \quad r = 1, 2, \ldots, n \tag{7}$$

therefore:

$$\begin{gathered} a_1 = -s_1\,, \qquad 2!\ a_2 = (s_1)^2 - s_2\,, \qquad 3!\ a_3 = -(s_1)^3 + 3\,s_1\,s_2 - 2\,s_3\,, \\ 4!\ a_4 = (s_1)^4 - 6\,(s_1)^2\,s_2 + 8\,s_1\,s_3 + 3\,(s_2)^2 - 6\,s_4\,, \quad etc. \end{gathered} \tag{8}$$

in particular, det $\mathbf{A} = (-1)^n\,a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that $\mathbf{A}$ and its transpose have the same determinant. In [20, 21] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & \cdots & 0 \\ s_2 & s_1 & k-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_{k-2} & \cdots & \cdots & 1 \\ s_k & s_{k-1} & \cdots & \cdots & s_1 \end{vmatrix}, \qquad k = 1, \dots, n. \tag{9}$$

The Faddeev-Sominsky's procedure [13-16, 19, 22] to obtain $\mathbf{A}^{-1}$ is a sequence of algebraic computations on the powers $\mathbf{A}^r$ and their traces, in fact, this algorithm is given via the instructions:

$$\begin{array}{lll} \mathbf{B}_1 = \mathbf{A}, & q_1 = \text{tr}\,\mathbf{B}_1, & \mathbf{C}_1 = \mathbf{B}_1 - q_1\,\mathbf{I}, \\ \mathbf{B}_2 = \mathbf{C}_1\,\mathbf{A}, & q_2 = \frac{1}{2}\,\text{tr}\,\mathbf{B}_2, & \mathbf{C}_2 = \mathbf{B}_2 - q_2\,\mathbf{I}, \\ \vdots & \vdots & \vdots \\ \mathbf{B}_{n-1} = \mathbf{C}_{n-2}\,\mathbf{A}, & q_{n-1} = \frac{1}{n-1}\,\text{tr}\,\mathbf{B}_{n-1}, & \mathbf{C}_{n-1} = \mathbf{B}_{n-1} - q_{n-1}\,\mathbf{I}, \\ \mathbf{B}_n = \mathbf{C}_{n-1}\,\mathbf{A}, & q_n = \frac{1}{n}\,\text{tr}\,\mathbf{B}_n, & \mathbf{C}_n = \mathbf{B}_n - q_n\,\mathbf{I} = \mathbf{0}, \end{array} \tag{10}$$

then:

$$\mathbf{A}^{-1} = \frac{1}{q_n}\mathbf{C}_{n-1}. \tag{11}$$

For example, if we apply (10) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (11) reproduces (3). By mathematical induction one can prove that (10) and (11) are equivalent to (3), (4) and (5), showing [16] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (10) we can see that [22]:

$$\mathbf{C}_k = \mathbf{A}^k + a_1\mathbf{A}^{k-1} + a_2\mathbf{A}^{k-2} + \dots + a_{k-1}\mathbf{A} + a_k\mathbf{I}, \qquad k = 1, 2, \dots, n-1, \tag{12}$$

and for $k = n - 1$:

$$\mathbf{C}_{n-1} = \mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + a_2\mathbf{A}^{n-3} + \dots + a_{n-2}\mathbf{A} + a_{n-1}\mathbf{I} \overset{(3)}{=} -a_n\mathbf{A}^{-1},$$

in harmony with (11) because $a_n = -q_n$. The property $\mathbf{C}_n = \mathbf{0}$ is equivalent to (2). If $\mathbf{A}$ is singular, the process (10) gives the adjoint matrix of $\mathbf{A}$ [2, 3], in fact, $Adj\,\mathbf{A} = (-1)^{n+1}\mathbf{C}_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of $\mathbf{A}$ [6]:

$$\mathbf{A}\,\vec{u}_k = \lambda_k\,\vec{u}_k, \qquad k = 1, 2, \dots, n, \tag{13}$$

because for a given value of k, each column of:

$$\mathbf{Q}_k \equiv \lambda_k^{n-1}\,\mathbf{I} + \lambda_k^{n-2}\,\mathbf{C}_1 + \cdots + \mathbf{C}_{n-1}\,, \tag{14}$$

satisfies (13), and therefore all columns of $\mathbf{Q}_k$ are proportional to each other, that is, $rank\ \mathbf{Q}_k = 1$ [19, 23].

3. AN ALTERNATIVE TO GOWER'S PSEUDOINVERSE

Here we consider the situation $1 \le p \le (n-1)$ with the condition $a_p \neq 0$, thus the multiplicity of the eigenvalue zero is $(n-p)$ and from (1) we deduce that [19]:

$$a_j = 0, \qquad j = p+1, \dots, n, \tag{15}$$

$$\lambda_k^p + a_1\,\lambda_k^{p-1} + \cdots + a_{p-1}\,\lambda_k + a_p = 0, \qquad k = 1, \dots, p, \tag{16}$$

which allows to show the property:

$$\mathbf{A}^{p+1} + a_1\,\mathbf{A}^p + \cdots + a_{p-1}\,\mathbf{A}^2 + a_p\,\mathbf{A} = \mathbf{0}, \tag{17}$$

if **A** is non-defective, that is, has *n* independent eigenvectors.

From (10) and (12) we observe that (17) means:

$$\mathbf{C}_p\,\mathbf{A} = \mathbf{B}_{p+1} = \mathbf{0}, \tag{18}$$

that is:

$$\mathbf{B}_j = \mathbf{C}_j = \mathbf{0}, \qquad j = p+1, \dots, n. \tag{19}$$

Thus, from (18) we have that:

$$\mathbf{0} = \mathbf{A}\,\mathbf{C}_p = \mathbf{A}\left(\mathbf{B}_p + a_p\,\mathbf{I}\right) = \mathbf{A}\,\mathbf{C}_{p-1}\,\mathbf{A} + a_p\,\mathbf{A} \qquad \therefore \qquad \mathbf{A}\left(-\frac{1}{a_p}\,\mathbf{C}_{p-1}\right)\mathbf{A} = \mathbf{A},$$

whose comparison with (4) gives the inverse matrix:

$$\mathbf{A}^- = -\frac{1}{a_p}\,\mathbf{C}_{p-1}\,, \tag{20}$$

therefore:

$$\mathbf{A}\,\mathbf{A}^- = \mathbf{A}^-\,\mathbf{A} = -\frac{1}{a_p}\,\mathbf{B}_p\,. \tag{21}$$

Hence from (4), (12) and (20):

$$\mathbf{A}^{-}\,\mathbf{A}\,\mathbf{A}^{-} = -\frac{1}{a_p}\,\mathbf{A}^{-}\,\mathbf{A}\left(\mathbf{A}^{p-1} + a_1\,\mathbf{A}^{p-2} + \cdots + a_{p-2}\,\mathbf{A} + a_{p-1}\,\mathbf{I}\right) = -\frac{1}{a_p}\,\mathbf{C}_{p-1} = \mathbf{A}^{-}\,,$$

because $\mathbf{A}^{-}\mathbf{A}\;\mathbf{A}^{k} = \mathbf{A}^{k}$, then (20) verifies (5).

We must note that Gower employs the inverse:

$$\mathbf{A}_G^{-} = -\frac{1}{a_p}\left(\mathbf{C}_{p-2} - \frac{a_{p-1}}{a_p}\,\mathbf{C}_{p-1}\right)\mathbf{A} = -\frac{1}{a_p}\left(\mathbf{B}_{p-1} - \frac{a_{p-1}}{a_p}\,\mathbf{B}_p\right), \qquad (22)$$

with the properties (4) and (5), but is immediate to see that:

$$\mathbf{A}_G^{-}\,\mathbf{A} = \mathbf{A}^{-}\,\mathbf{A}. \qquad (23)$$

We consider that our inverse (20) is more simple and natural than (22) because it was implied by the Faddeev-Sominsky's algorithm [13-16, 19, 22, 24].

4. CONCLUSIONS

We emphasize that (20) was obtained under the constraints $1 \leq p \leq (n-1)$, $a_p \neq 0$ and **A** non-defective. The Leverrier-Faddeev's technique, according to Householder [24, 25], was rediscovered and improved by Souriau [26] and Frame [27]. It is interesting to mention that the method (10) was successfully applied [28-30] in general relativity to study the embedding of spacetimes into pseudo-Euclidean spaces.

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