ABSTRACT

We employ the generalized inverse matrix of Moore-Penrose to study the existence and uniqueness of the solutions for over- and under-determined linear systems, in harmony with the least squares method.

Keywords: Linear systems, SVD, Least squares technique, Pseudoinverse of Moore-Penrose

1. INTRODUCTION

For any real matrix $A_{nxm}$, Lanczos [1, 2] introduces the matrix:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A^T \\ A^T & 0 \end{pmatrix},$$

with $A^T$ denoting the transpose matrix, and studies the eigenvalue problem:
where the proper values are real because $S$ is a real symmetric matrix. Besides:

$$\text{rank } A \equiv p = \text{Number of positive eigenvalues of } S,$$

such that $1 \leq p \leq \min(n, m)$. Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, \ldots, \lambda_p, -\lambda_1, -\lambda_2, \ldots, -\lambda_p, 0, 0, \ldots, 0,$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of $S$, named ‘essential axes’ by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}^n / m,$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{n \times m} \vec{v}_{mx1} = \lambda \vec{u}_{nx1}, \quad A^T_{mxn} \vec{u}_{nx1} = \lambda \vec{v}_{mx1},$$

hence:

$$A^T A \vec{v} = \lambda^2 \vec{v}, \quad A A^T \vec{u} = \lambda^2 \vec{u},$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{n \times p} = \begin{pmatrix} \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \end{pmatrix}, \quad V_{m \times p} = \begin{pmatrix} \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \end{pmatrix},$$

verifying $U^T U = V^T V = I_{p \times p}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk},$$

therefore $\vec{u}_j \cdot \vec{u}_k = 2\delta_{jk}$, $j, k = 1, 2, \ldots, p$. Thus, the Singular Value Decomposition (SVD) express [1-5] that $A$ is the product of three matrices:

$$A_{n \times m} = U_{n \times p} \Lambda_{p \times p} V^T_{p \times m}, \quad \Lambda = \text{Diag} \left( \lambda_1, \lambda_2, \ldots, \lambda_p \right).$$

This relation tells that in the construction of $A$ we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to $A$. Golub [6] mentions that the SVD has played a very important role in computations, in solving least squares problems [7], in signal processing problems, and so on; it is just a very simple decomposition, yet it is of fundamental importance in many problems arising in technology.
It is important to observe that the symmetric matrices \((UU^T)_{n \times n}\) and \((VV^T)_{m \times m}\) are identity matrices for arbitrary vectors into their respective spaces of activation [5], that is:
\[
UU^T \vec{u} = \vec{u}, \quad \forall \ \vec{u} \in Col U, \quad VV^T \vec{v} = \vec{v}, \quad \forall \ \vec{v} \in Col V; \quad (11)
\]
because, (10) allows obtain the SVD of the Gram matrices:
\[
(AA^T)_{n \times n} = U \Lambda^2 U^T, \quad (A^T A)_{m \times m} = V \Lambda^2 V^T, \quad (12)
\]
such that \(p = \text{rank} A = \text{rank} (AA^T) = \text{rank} (A^T A)\). From (10) and (12) we observe that:
\[
\text{Col} A = \text{Col} (AA^T) = \text{Col} U, \quad \text{Col} A^T = \text{Col} (A^T A) = \text{Col} V. \quad (13)
\]

The eigenvectors associated with \(\lambda = 0\) verify the equations:
\[
A \vec{v}_j^0 = \vec{0}, \quad j = 1, \ldots, m - p, \quad A^T \vec{u}_k^0 = \vec{0}, \quad k = 1, \ldots, n - p,
\]
\[
\vec{v}_r \cdot \vec{v}_j^0 = 0, \quad \forall \ r, j, \quad \vec{u}_k \cdot \vec{u}_k^0 = 0, \quad \forall \ t, k \quad (14)
\]
therefore:
\[
V^T \vec{v}_j^0 = \vec{0}, \quad \forall \ j, \quad U^T \vec{u}_k^0 = \vec{0}, \quad \forall \ k, \quad (15)
\]
\[
A \vec{x} \in \text{Col} U \text{ and } A^T A \vec{x} \in \text{Col} V, \quad \forall \ \vec{x} \in \mathbb{E}^m, \\
A^T \vec{y} \in \text{Col} V \text{ and } AA^T \vec{y} \in \text{Col} U, \quad \forall \ \vec{y} \in \mathbb{E}^n.
\]

In Sec. 2 we exhibit the Moore-Penrose’s pseudoinverse of \(A\) [8-13] via the corresponding SVD [14-16], which is useful in Sec. 3 to study the solutions of over- and under-determined linear systems [2, 5] in the spirit of the least squares method [7, 17].

2. GENERALIZED INVERSE

The Moore-Penrose’s inverse [2, 8-13] is given by:
\[
A_{m \times n}^+ = V_{m \times p} \Lambda_{p \times p}^{-1} U_{p \times n}^T, \quad (16)
\]
which coincides with the natural inverse obtained by Lanczos [2, 5]. The matrix (16) satisfies the relations [10, 11, 13]:
\[
AA^+ A = A, \quad A^+ A A^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+ A)^T = A^+ A, \quad (17)
\]
that characterize the pseudoinverse of Moore-Penrose. In particular, from (10), (11) and (16):
\[ A A^+ = U U^T \quad \therefore \quad A A^+ \bar{u} = \bar{u}, \quad \forall \; \bar{u} \in \text{Col} \; U, \quad (18) \]

\[ A^+ A = V V^T \quad \therefore \quad A^+ A \bar{v} = \bar{v}, \quad \forall \; \bar{v} \in \text{Col} \; V. \]

The use of (8) and (10) into (16) implies the following expression for the Lanczos generalized inverse:

\[ A^+ = (\bar{t}_1 \bar{t}_2 \cdots \bar{t}_n), \quad \bar{t}_j = \frac{u^{(j)}_1}{\lambda_1} \bar{v}_1 + \frac{u^{(j)}_2}{\lambda_2} \bar{v}_2 + \cdots + \frac{u^{(j)}_p}{\lambda_p} \bar{v}_p, \quad j = 1, \ldots, n, \quad (19) \]

where \( u^{(j)}_k \) means the \( j \)th component of \( \bar{u}_k \); similarly:

\[ (A^+)^T = (\bar{r}_1 \bar{r}_2 \cdots \bar{r}_m), \quad \bar{r}_k = \frac{v^{(k)}_1}{\lambda_1} \bar{u}_1 + \frac{v^{(k)}_2}{\lambda_2} \bar{u}_2 + \cdots + \frac{v^{(k)}_p}{\lambda_p} \bar{u}_p, \quad k = 1, \ldots, m, \quad (20) \]

therefore:

\[ \text{Col} \; A^+ = \text{Col} \; V, \quad \text{Col} \; (A^+)^T \equiv \text{Col} \; (U \Lambda^{-1} V^T) = \text{Col} \; U. \quad (21) \]

We can use (16) to construct the pseudoinverse of each Gram matrix, in fact [13]:

\[ (A^T A)^+_{m \times m} = V \Lambda^{-2} V^T, \quad (A A^T)^+_{n \times n} = U \Lambda^{-2} U^T, \quad (22) \]

with the interesting properties:

\[ (A^T A)^+ A^T = A^+, \quad (A A^T)^+ A = (A^+)^T, \quad (A^T A)^+(A^T A) = A^+ A = V V^T. \quad (23) \]

Each matrix has a unique inverse because every matrix is complete within its own spaces of activation. The activated \( p \)-dimensional subspaces (eigenspaces / operational spaces) are uniquely associated with the given matrix [5].

### 3. LINEAR SYSTEMS

We want to find \( \bar{x} \in E^m \) verifying the linear system:

\[ A \bar{x} = \bar{b}, \quad (24) \]

for the data \( A_{n \times m} \) and \( \bar{b} \in E^n \). It is convenient to consider two situations:

**a). Over-determined linear system** [2,5]: In this case we have more equations than unknowns, that is, \( m < n \).

Lanczos [18] comments that the ingenious method of least squares makes it possible to adjust an arbitrarily over-determined and incompatible set of equations. The problem of
minimizing \((A\bar{x} - \bar{b})^2\) has always a definite solution, no matter how compatible or incompatible the given system is. The least square solution of (24) satisfies \([5, 17]\):

\[ A^T A \bar{x} = A^T \bar{b}, \quad \bar{x} \in \text{Col} V, \quad p = m, \tag{25} \]

and the remarkable fact about (25) is that it always gives an even-determined (balanced) system, no matter how strongly over-determined the original system has been.

The system (25) is compatible because from (13) and (15) we have that \(A^T \bar{b}\) is into \(\text{Col} (A^T A) = \text{Col} V\). Now we multiply (25) by \((A^T A)^+\) and we use (11) and (23) to obtain the solution:

\[ \bar{x} = A^+ \bar{b}, \tag{26} \]

which is unique because \(p = m\), that is, \(\text{Col} V = E^m\), then in (14) the system \(A\bar{x}_j^\circ\) only has the trivial solution; hence the Moore-Penrose’s least square solution of (24). The expression (26) is in harmony with the results in \([19-22]\).

We have eliminated over-determination (and possibly incompatibility) by the method of multiplying both sides of (24) by \(A^T\). The unique solution thus obtained coincides with the solution generated with the help of \(A^+ [5]\).

b). Under-determined linear system \([2, 5]\): There are more unknowns than equations, that is, \(n < m\).

In this case we may try the least square formulation of (24), that is, to accept (26), however, now the solution is not unique because \(p < m\) and the system \(A\bar{x}_j^\circ\) has \(m - p\) non-trivial independent solutions; an under-determined system remains thus under-determined, even in the least square approach.

An alternative process is to transform the original \(\bar{x}\) into the new unknown \(\bar{z}\) via the relation \([5]\):

\[ \bar{x} = A^T \bar{z}, \tag{27} \]

then (24) acquires the structure \(AA^T \bar{z} = \bar{b}\) whose least square solution is given by the pseudoinverse of Moore-Penrose:

\[ \bar{z} = (AA^T)^+ \bar{b} + \sum_{j=1}^{n-p} c_j \bar{z}_j^\circ, \tag{28} \]

where the quantities \(c_j\) are arbitrary and the \(\bar{z}_j^\circ\) are \(n - p\) independent vectors generating the \(\text{Kernel} (AA^T) = \text{Kernel} (A^T)\) \([13]\), that is:

\[ A^T \bar{z}_j^\circ = \vec{0}, \quad j = 1, ..., n - p. \tag{29} \]

Thus, from (16), (22), (23), (28) and (29) we have that the solution of (27) is given by:
\[ \hat{x} = A^T (AA^T)^+ \hat{b} = V \Lambda^{-1} U^T \hat{b} = A^+ \hat{b}, \]

in agreement with (26).

Although that (26) is not unique for the under-determined case, we can say that it is the ‘natural solution’ for the linear system (24).

### 4. CONCLUSIONS

Our study shows the importance of the SVD [1-6, 14-16] of a matrix and of the corresponding Moore-Penrose’s inverse [8-13], to elucidate the least square solution [7, 17-22] for over- and under-determined linear systems [2, 5].

### References


