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SHORT COMMUNICATION

Stirling numbers with negative indices

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ABSTRACT

We realize applications of expressions for Stirling numbers with negative indices, in particular, we show that the formulas of Zhi-Hong Sun are consequences of the known identities of Schläfli and Gould.

***Keywords*:** Gould and Schläfli formulas, Stirling numbers

1. INTRODUCTION

Knuth [1] comments that Gould was the first to extend the domain of Stirling numbers to negative values of the indices, via the duality properties [2]:

$$S_{-n}^{(-N)} = (-1)^{N+n} \; S_N^{[n]} \, , \tag{1}$$

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$$S_{-n}^{[-N]} = (-1)^{N+n}\ S_N^{(n)}\ , \tag{2}$$

where: $S_r^{(j)}$ and $S_q^{[t]}$ represent the Stirling numbers of the first and second kind, respectively [2].

In Sec. 2 we make several applications of (1) and (2), in particular, we show that the identities of Schläfli [2-5] and Gould [2, 5, 6] imply the Sun's formulas [7] for Stirling numbers.

2. SUN'S RELATIONS

Sun [7] used the inversion method to deduce the following expressions:

$$S_{m+n}^{[n]} = \sum_{k=0}^{m} \binom{m-n}{m-k}\binom{m+n}{m+k} S_{m+k}^{[k]}\ , \tag{3}$$

$$S_{m+n}^{(n)} = \sum_{k=0}^{m} \binom{m-n}{m-k}\binom{m+n}{m+k} S_{m+k}^{(k)}\ . \tag{4}$$

On the other hand, we have the Schläfli's relation [2-5]:

$$S_n^{(n-m)} = (-1)^m\ \sum_{k=0}^{m} \binom{m+n}{m-k}\binom{m-n}{m+k} S_{m+k}^{[k]}\ , \tag{5}$$

where we realize the change $n \to -n$ to obtain:

$$S_{-n}^{(-(m+n))} = (-1)^m\ \sum_{k=0}^{m} \binom{m-n}{m-k}\binom{m+n}{m+k} S_{m+k}^{[k]}\ ,$$

which implies (3) through (1).

The Gould's identity is given by [2, 5, 6]:

$$S_n^{[n-m]} = (-1)^m\ \sum_{k=0}^{m} \binom{m+n}{m-k}\binom{m-n}{m+k} S_{m+k}^{(k)}\ , \tag{6}$$

where we make the transformation $n \to -n$ to deduce:

$$S_{-n}^{[-(m+n)]} = (-1)^m\ \sum_{k=0}^{m} \binom{m-n}{m-k}\binom{m+n}{m+k} S_{m+k}^{(k)}\ ,$$

which is equivalent to (4) due to (2).

We have the recurrence relation [2]:

$$S_n^{[k]} = S_{n-1}^{[k-1]} + k\ S_{n-1}^{[k]}\ , \tag{7}$$

where we apply the changes $k \to -k$ and $n \to -n$ to obtain:

$$S_{-n}^{[-k]} = S_{-(n+1)}^{[-(k+1)]} - k\, S_{-(n+1)}^{[-k]} \quad .$$

and with the help of (2) we deduce the known recurrence relation [2, 8]:

$$S_{k+1}^{(n+1)} = S_{k}^{(n)} - k\, S_{k}^{(n+1)} . \tag{8}$$

The following expression is the identity (9.7) into [2]:

$$S_{n}^{[m]} = \sum_{k=0}^{r} (m-k)\, S_{n-k-1}^{[m-k]} + S_{n-r-1}^{[m-r-1]} , \tag{9}$$

where we can make the transformations $m \to -m$ and $n \to -n$, and to employ (2) to obtain its companion relation:

$$S_{m}^{(n)} = \sum_{k=0}^{r} (m+k)\, S_{m+k}^{(n+k+1)} + S_{m+r+1}^{(n+r+1)} . \tag{10}$$

Similarly, we have the Kramp's formula [1, 2, 9]:

$$n\, S_{n}^{(n-k)} = (-1)^{k} \sum_{j=0}^{k} (-1)^{j} \binom{n-j}{k+1-j} S_{n}^{(n-j)}, \tag{11}$$

where we apply $n \to -n$ and we use (1) to deduce the property:

$$n\, S_{n+k}^{[n]} = \sum_{r=0}^{k} (-1)^{r} \binom{n+k}{r+1} S_{n+k-r}^{[n]} . \tag{12}$$

Comtet [10] gives the following values:

$$S_{n}^{(n-1)} = -\binom{n}{2}, \quad S_{n}^{(n-2)} = \frac{1}{4}\binom{n}{3}(3n-1), \quad S_{n}^{(n-3)} = -\frac{1}{2}\binom{n}{4} n(n-1), \tag{13}$$

then with (1), (13) and the identity [2, 11]:

$$\binom{-m}{j} = (-1)^{j} \binom{m+j-1}{j}, \tag{14}$$

we find the expressions [1, 2, 12]:

$$S_{n+1}^{[n]} = \binom{n+1}{2}, \qquad S_{n+2}^{[n]} = \frac{1}{4}\binom{n+2}{3}(3n+1),$$
$$S_{n+3}^{[n]} = \frac{1}{2}\binom{n+3}{4} n(n+1), \tag{15}$$

Sánchez-Peregrino [13] used induction to show the result:

$$S_{m+n}^{[k]} = \sum_{r=0}^{n} \sum_{j=0}^{n} (-1)^{n+r+j} \binom{n}{j} k^j \, S_{n-j}^{[r]} \, S_m^{[k-r]} \,, \qquad (16)$$

where the changes $k \to -k$, $m \to -m$ and (2) imply the identity:

$$S_k^{(m-n)} = \sum_{r=0}^{n} \sum_{j=0}^{n} \binom{n}{j} k^j \, S_{n-j}^{[r]} \, S_{k+r}^{(m)} \,, \qquad m \geq n. \qquad (17)$$

3. CONCLUSIONS

Thus, we see that the expressions (3) and (4) found by Sun [7] via inversion technique, are implied by the formulas of Schläfli and Gould, which shows the usefulness of the relations (1) and (2). The authors Gupta [14, 15] and Gessel-Stanley [16] also obtained the duality properties (1) and (2), respectively.

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