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Bayesian and Non Bayesian Method of Estimation of Scale Parameter of Gamma Distribution under Symmetric and Asymmetric Loss Functions

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ABSTRACT

In this investigation, we consider Bayesian and Non-Bayesian estimation problems of unknown scale parameter of Gamma distribution assuming the shape parameter as known and derive Bayes and Classical (Non-Bayes) estimators of the scale parameter. Bayes estimators are obtained under symmetric (squared error) and asymmetric (linex and precautionary) loss functions using a non-informative prior. The risk efficiency of Bayes estimators is also obtained under these loss functions. Finally, the simulation study is done to compare the performance of these estimators using MATLAB software.

Keywords**:** Bayes Estimator, Squared Error Loss Function, Linex Loss Function, Precautionary Loss Function, Risk Function

1. INTRODUCTION

In Reliability studies, the most popular models which are used in life testing include the Exponential and Gamma distribution. These distributions posses both scale and shape parameter and hence, found to be quite flexible to analyze any positive real data. Gamma distribution has wide applicability in real life, e.g., in life testing the waiting time until death

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is a random variable that is frequently modeled with Gamma distribution. Also, in reliability, the arrival time in the Poisson process has gamma distribution. The probability density function (p.d.f) of Gamma distribution with two parameters α and λ is

$$f(x;\alpha,\lambda)=\frac{x^{\alpha-1}\exp(-x/\lambda)}{\Gamma\alpha\ \lambda^{\alpha}},\ x\geq 0;\ \lambda\geq 0;\ \alpha\geq 0. \tag{1}$$

where: α is the shape parameter which is known and λ is scale parameter which is unknown. The Gamma distribution reduces to some other distributions. When $\lambda=1$ it is the simple (or Standard) gamma density, when $\alpha=1$, it is exponential distribution and when $\alpha=n$ is an integer, it is called Erlang distribution. The Gamma distribution has been extensively used in the areas of Reliability, Insurance, Life testing Meteorology, Climatology and many other physical situations see (Sinha and Kale (1980); Von Alven (1964)). Basu and Ebrahimi (1991) provided the Bayesian approach to life testing and reliability estimation using asymmetric loss function. Takada and Nagata (1995) proposed a fixed-width sequential confidence interval for the mean of a gamma distribution and Norstrom (1996) provided the use of precautionary loss function in risk analysis.

Farsipour and Zakerzadeh (2005) have provided Bayes estimate of scale parameter of Gamma distribution under asymmetric squared error loss function with restriction to the principle of invariance and risk unbiasedness. Bayesian and Non-Bayesian estimation of shape parameter, reliability, and failure rate function of exponentiated Gamma distribution under linex and squared error loss function was studied by Shawky and Bakban (2008).

Dey (2010) studied Bayes estimates of the shape parameter of the Generalised Exponential Distribution under different loss functions.

Two-stage sequential procedures were developed for fixed-width interval estimation of the scale parameter in a Gamma distribution by Zacks and Khan (2011) and exact properties were obtained for the two-stage procedure and some asymptotics and approximations were given for the operating characteristics of the sequential procedure with some numerical computations. Singh *et al.*, (2011) studied Bayes estimate of parameters and reliability function of Exponentiated Exponential distribution under general entropy loss function for the type-II censored sample. Dey (2012) studied Bayes estimate of the unknown parameter and reliability function of an Inverse Rayleigh distribution using symmetric squared error loss function and asymmetric linear exponential loss function using a non-informative prior. All these papers made a good contribution in the field of Bayesian analysis.

In this investigation, our aim is to obtain classical estimator i.e maximum likelihood estimator and Bayes estimator of the scale parameter of Gamma setup under symmetric and asymmetric loss function and to make some comparisons. The performance of the estimators is assessed on the basis of their relative risk under the three types of loss functions. Here, we estimate scale parameter of Gamma distribution as scale parameter determines the risk or statistical dispersion of the probability distribution. The main motive behind obtaining Bayes estimators of scale parameter of Gamma distribution is wide applicability of Gamma distribution in reliability theory.

Also, Bayesian analysis has been found useful in reliability theory as it provides inference with the increased quality provided the prior information is chosen accurately and also it requires less sample size as compared to the classical estimator.

The remaining contents of this research article are organized in different sections as follows. Section 2, introduces the prior distributions and the loss functions used in the paper. In, Section 3, we obtain Non-Bayesian estimator i.e Maximum likelihood estimator and the Bayes estimator of scale parameter under three different loss functions. The associated risk functions and risk efficiency of Bayes estimators under different loss functions are given in section 4. In, Section 5, numerical simulation is carried out to examine the statistical performance of Bayes estimates and Mle. Finally, in last Section 6, we conclude our investigation by summarizing the research work and highlighting the noble features of investigation done.

2. PRIOR DISTRIBUTIONS AND LOSS FUNCTIONS APPROACH FOR BAYESIAN ANALYSIS

An appropriate choice of prior(s) is indispensable for Bayesian analysis. However, very often, researchers select prior(s) according to their subjective knowledge and beliefs. Nevertheless, if we have enough information about the parameter(s) we should use informative prior(s), otherwise it is better to consider non-informative prior(s) or vague prior(s). In this paper, we consider Jeffrey's non-informative prior to the scale parameter λ which is $g(\lambda)\propto\sqrt{I(\lambda)}$, where $I(\lambda)$ is a $k\times k$ Fisher's (information) matrix whose (i,j)th element is $-E\left[\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\log L\left(\underline{x}/\lambda\right)\right]$. Let $\underline{x}=(x_1,x_2,x_3,...,x_n)$ is a random sample from Gamma distribution. Then likelihood function of given sample observations is

$$L\left(\frac{\underline{x}}{\lambda}\right)=\frac{\prod_{i=1}^{n}x_i^{\alpha-1}\exp\left(-\sum x_i/\lambda\right)}{(\Gamma\alpha)^n\lambda^{n\alpha}} \tag{2}$$

and

$$-E\left[\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\log L\left(\underline{x}/\lambda\right)\right]=\frac{n\alpha}{\lambda^2}$$

$$\Rightarrow\quad g(\lambda)\propto\sqrt{\frac{n\alpha}{\lambda^2}}$$

$$\Rightarrow\quad g(\lambda)\propto\left(\frac{1}{\lambda}\right) \tag{3}$$

From a decision-theoretic viewpoint, in order to select the best estimator, a loss function must be specified and is used to represent a penalty associated with each of the possible estimates. Since, there is no specific analytic procedure that allows us to identify the appropriate loss function to be used, customarily, in most cases for convenience, researchers use the squared error loss function which is symmetrical, and associates equal importance to the losses due to overestimation and underestimation of equal magnitude and obtain the posterior mean as the Bayesian estimate. No doubt, the use of squared error loss function is well justified when the loss is symmetric in nature. However, for some estimation and prediction problems, the real loss is often asymmetric. Thus in order to make the statistical inferences more practical and applicable, we often needs to choose an asymmetric loss function. One of the most popular is the Linex loss function which was introduced by Varian (1975), and popularized by Zellner (1986). Norstrom (1996) was the first to introduce the precautionary loss function which is asymmetric in nature.

(a) The squared error loss function is given by

$$L\left(\hat{\lambda},\lambda\right)=\left(\hat{\lambda}-\lambda\right)^2 \tag{4}$$

which is symmetric and where $\hat{\lambda}$ is an estimate of λ.

(b) The Linex loss function is given by

$$L(\sigma)=\exp(a\sigma)-a\sigma-1;\ a\neq 0 \tag{5}$$

where $\sigma=\frac{\hat{\lambda}}{\lambda}-1$ and $\hat{\lambda}$ is an estimate of λ. Here a represents the shape parameter of the loss function. The behavior of the linex loss function changes with the choice of a. For $a > 0$, over estimation is considered to be more serious than under estimation and for $a<0$, under estimation is considered to be more serious than over estimation.

(c) The Precautionary loss function is given by

$$L\left(\hat{\lambda},\lambda\right)=\frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}} \tag{6}$$

The Bayes estimator under Precautionary loss function is obtained by solving the equation $\hat{\lambda}=\sqrt{E\left(\lambda^2/X\right)}$.

This loss function is useful when underestimation is more serious than overestimation.

3. NON-BAYESIAN AND BAYESIAN ESTIMATORS OF SCALE PARAMETER

When a random sample $\underline{x} = (x_1, x_2, x_3, ..., x_n)$ is obtained from Gamma distribution with p.d.f given by (1), the likelihood function is given by

$$L(\underline{x}, \lambda) = \frac{\prod_{i=1}^{n} x_i^{\alpha-1} \exp\left(-\sum x_i / \lambda\right)}{(\Gamma\alpha)^n \lambda^{n\alpha}},$$

and the logarithm of the likelihood function is given by

$$\log L = \log \prod_{i=1}^{n} x_i^{\alpha-1} - \frac{\sum x_i}{\lambda} - \log(\Gamma\alpha)^n - n\alpha \log \lambda,$$

and hence the maximum likelihood estimator of λ, denoted by $\hat{\lambda}_{mle}$, is given by

$$\hat{\lambda}_{mle} = \frac{\sum x_i}{n\alpha} \tag{7}$$

and for the Bayesian estimators combining the likelihood function (2) with the prior distribution (3) and using Bayes theorem, we get the posterior density of λ:

$$P(\lambda/x) = \frac{\exp\left(-\sum \frac{x_i}{\lambda}\right)\left(\sum x_i\right)^{n\alpha}}{\Gamma n\alpha \ \lambda^{n\alpha+1}}; \ x \geq 0; \ \lambda \geq 0; \ \alpha \geq 0. \tag{8}$$

- **Bayes Estimator of λ Based on Squared Error Loss Function**

Under squared error loss function $L(\hat{\lambda}, \lambda) = (\hat{\lambda} - \lambda)^2$ the Baye's estimator is obtained by minimizing $E\left[L(\hat{\lambda}, \lambda)\right] = \int_0^\infty (\hat{\lambda} - \lambda)^2 P\left(\frac{\lambda}{x}\right) d\lambda = \hat{\lambda}^2 + \frac{\sum x_i^2}{n\alpha(n\alpha - 1)} - \frac{2\hat{\lambda}\sum x_i}{(n\alpha - 1)}$.

The Bayes estimator of λ denoted by $\hat{\lambda}_{SB}$ is the posterior mean given by

$$\hat{\lambda}_{SB} = \frac{\sum x_i}{n\alpha - 1} \tag{9}$$

- **Bayes Estimator of λ Based on Linex Loss Function**

Under the Linex loss function (5), the posterior expected loss is given by

$$E\left[L\left(\hat{\lambda},\lambda\right)\right]=\int_0^\infty\left\{\exp\left[a\left(\frac{\hat{\lambda}}{\lambda}-1\right)-a\left(\frac{\hat{\lambda}}{\lambda}-1\right)-1\right]P\left(\lambda/x\right)\right\}d\lambda=\frac{\left(\sum x_i\right)^{n\alpha}}{\left(\sum x_i-a\hat{\lambda}\right)^{n\alpha}}\exp\left(-a\right)-a\frac{\hat{\lambda}n\alpha}{\sum x_i}+a-1.$$

The Bayes estimator of λ under the linex loss function denoted by $\hat{\lambda}_{LB}$ is given by

$$\hat{\lambda}_{LB}=\frac{\sum x_i\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}{a} \tag{10}$$

- **Bayes Estimator of λ Based on Precautionary Loss Function**

Under the Precautionary loss function (6), the posterior expectation is given by

$$E\left[L\left(\hat{\lambda},\lambda\right)\right]=\int_0^\infty\frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}}P\left(\lambda/x\right)d\lambda=\hat{\lambda}+\frac{1}{\hat{\lambda}}\frac{\left(\sum x_i\right)^2}{\left(n\alpha-1\right)\left(n\alpha-2\right)}-\frac{2\sum x_i}{n\alpha-1}.$$

The Bayes estimator of λ under the Precautionary loss function denoted by $\hat{\lambda}_{PB}$ is given by

$$\hat{\lambda}_{PB}=\frac{\sum x_i}{\sqrt{\left(n\alpha-1\right)\left(n\alpha-2\right)}} \tag{11}$$

4. RISK FUNCTIONS AND RISK EFFICIENCY OF BAYES ESTIMATORS

In decision theoretic approach, a good decision function is one that has a small value of the risk function. The risk function is defined as

$$R\left(\hat{\lambda},\lambda\right)=E\left[L\left(\hat{\lambda},\lambda\right)\right]=\int L\left(\hat{\lambda},\lambda\right)f\left(x,\lambda\right)dx.$$

The likelihood function of the random sample $\left(x_1,x_2,...,x_n\right)$ is given by

$$L\left(\underline{x},\lambda\right)=\frac{\prod_{i=1}^{n}x_i^{\alpha-1}\exp\left(-\sum x_i/\lambda\right)}{\left(\Gamma\alpha\right)^n\lambda^{n\alpha}}.$$

Let $S=\sum x_i$ and since $x_i,\ i=1,2,...,n$ are independent and identically distributed random variables from Gamma distribution with parameters (α,λ) then $S = \sum x_i$ ~ G(nα, λ). Thus p.d.f of S is given as $h(s)=\frac{s^{n\alpha-1}\exp(-s/\lambda)}{\Gamma n\alpha\ \lambda^{n\alpha}};\ s\geq 0;\ \lambda\geq 0$.

The risk function of the estimator $\hat{\lambda}_{SB}$ under squared error loss function is given by

$$R_S\left(\hat{\lambda}_{SB},\lambda\right)=E\left[L\left(\hat{\lambda}_{SB},\lambda\right)\right]=\int_0^\infty\left(\hat{\lambda}_{SB}-\lambda\right)^2.h(s)ds=\lambda^2\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}-\frac{2n\alpha}{(n\alpha-1)}+1\right] \quad (12)$$

The risk function of the estimator $\hat{\lambda}_{LB}$ under linex loss function is given by

$$R_L\left(\hat{\lambda}_{LB},\lambda\right)=E\left[L\left(\hat{\lambda}_{LB},\lambda\right)\right]=\int_0^\infty\left\{\exp\left[a\left(\frac{\hat{\lambda}_{LB}}{\lambda}-1\right)-a\left(\frac{\hat{\lambda}_{LB}}{\lambda}-1\right)-1\right]\right\}h(s)ds$$
$$=\exp\left(\frac{-a}{n\alpha+1}\right)-n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]+a-1 \quad (13)$$

The risk function of the estimator $\hat{\lambda}_{PB}$ under precautionary loss function is given by

$$R_P\left(\hat{\lambda}_{PB},\lambda\right)=E\left[L\left(\hat{\lambda}_{PB},\lambda\right)\right]=\int_0^\infty\frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}}h(s)ds=\lambda\left[\frac{n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}-2+\frac{\sqrt{(n\alpha-1)(n\alpha-2)}}{(n\alpha-1)}\right] \quad (14)$$

Following this, the risk efficiencies of risk functions (12), (13) and (14) are computed to check which estimator performs better under squared error, linex and precautionary loss function. A comparison of this type is needed to check whether an estimator is inadmissible under some loss function.

Risk Efficiency of $\hat{\lambda}_{LB}$ w.r.t $\hat{\lambda}_{SB},\hat{\lambda}_{PB}$ under Squared Error Loss Function

At this point our objective is to find risk function of the estimators $\hat{\lambda}_{SB},\hat{\lambda}_{LB},\hat{\lambda}_{PB}$ relative to squared error loss function and the risk function are given as:

$$R_S\left(\hat{\lambda}_{SB},\lambda\right)=\int_0^\infty\left(\hat{\lambda}_{SB}-\lambda\right)^2.h(s)ds=\lambda^2\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}-\frac{2n\alpha}{(n\alpha-1)}+1\right]$$

$$R_S\left(\hat{\lambda}_{LB},\lambda\right)=\int_0^\infty\left(\hat{\lambda}_{LB}-\lambda\right)^2.h(s)\,ds=\lambda^2\left[\frac{\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)^2(n\alpha)(n\alpha+1)}{a^2}+1-\frac{2\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)n\alpha}{a}\right]$$

In the same manner, we get

$$R_S\left(\hat{\lambda}_{PB},\lambda\right)=\int_0^\infty\left(\hat{\lambda}_{PB}-\lambda\right)^2.h(S)\,dS=\lambda^2\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)(n\alpha-2)}+1-\frac{2n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right]$$

The risk efficiency of $\hat{\lambda}_{LB}$ with respect to $\hat{\lambda}_{SB}$ under squared error loss function is defined as follows:

$$RE_S\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)=\frac{R_S\left(\hat{\lambda}_{LB},\lambda\right)}{R_S\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\left[\frac{\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)^2(n\alpha)(n\alpha+1)}{a^2}+1-\frac{2\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)n\alpha}{a}\right]}{\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}-\frac{2n\alpha}{(n\alpha-1)}+1\right]} \tag{15}$$

Similarly,

$$RE_S\left(\hat{\lambda}_{LB},\hat{\lambda}_{PB}\right)=\frac{R_S\left(\hat{\lambda}_{LB},\lambda\right)}{R_S\left(\hat{\lambda}_{PB},\lambda\right)}=\frac{\left[\frac{\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)^2(n\alpha)(n\alpha+1)}{a^2}+1-\frac{2\left(1-\exp\left(\frac{-a}{n\alpha+1}\right)\right)n\alpha}{a}\right]}{\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)(n\alpha-2)}+1-\frac{2n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right]} \tag{16}$$

$$RE_S\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)=\frac{R_S\left(\hat{\lambda}_{PB},\lambda\right)}{R_S\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)(n\alpha-2)}+1-\frac{2n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right]}{\left[\frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}-\frac{2n\alpha}{(n\alpha-1)}+1\right]} \tag{17}$$

Risk Efficiency of $\hat{\lambda}_{LB}$ w.r.t $\hat{\lambda}_{SB}, \hat{\lambda}_{PB}$ under Linex Loss Function

The risk function of the estimator $\hat{\lambda}_{LB}, \hat{\lambda}_{SB}, \hat{\lambda}_{PB}$ under linex loss function is given as:

$$R_L\left(\hat{\lambda}_{LB},\lambda\right)=\int_0^\infty\left\{\exp\left[a\left(\frac{\hat{\lambda}_{LB}}{\lambda}-1\right)-a\left(\frac{\hat{\lambda}_{LB}}{\lambda}-1\right)-1\right]\right\}h(S)\,dS=\exp\left(\frac{-a}{n\alpha+1}\right)-n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]+a-1.$$

$$R_L\left(\hat{\lambda}_{SB},\lambda\right)=\exp(-a)\left(1-\frac{a}{n\alpha-1}\right)^{-n\alpha}-\frac{a}{n\alpha-1}-1.$$

$$R_L\left(\hat{\lambda}_{PB},\lambda\right)=\exp(-a)\left(1-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right)^{-n\alpha}-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}n\alpha+a-1.$$

Thus, risk efficiencies are given as:

$$RE_L\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)=\frac{R_L\left(\hat{\lambda}_{LB},\lambda\right)}{R_L\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\exp\left(\frac{-a}{n\alpha+1}\right)-n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]+a-1}{\exp(-a)\left(1-\frac{a}{n\alpha-1}\right)^{-n\alpha}-\frac{a}{n\alpha-1}-1}. \qquad (18)$$

$$RE_L\left(\hat{\lambda}_{LB},\hat{\lambda}_{PB}\right)=\frac{R_L\left(\hat{\lambda}_{LB},\lambda\right)}{R_L\left(\hat{\lambda}_{PB},\lambda\right)}=\frac{\exp\left(\frac{-a}{n\alpha+1}\right)-n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]+a-1}{\exp(-a)\left(1-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right)^{-n\alpha}-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}n\alpha+a-1} \qquad (19)$$

$$RE_L\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)=\frac{R_L\left(\hat{\lambda}_{PB},\lambda\right)}{R_L\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\exp(-a)\left(1-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}\right)^{-n\alpha}-\frac{a}{\sqrt{(n\alpha-1)(n\alpha-2)}}n\alpha+a-1}{\exp(-a)\left(1-\frac{a}{n\alpha-1}\right)^{-n\alpha}-\frac{a}{n\alpha-1}-1} \qquad (20)$$

Risk Efficiency of $\hat{\lambda}_{LB}$ w.r.t $\hat{\lambda}_{SB}, \hat{\lambda}_{PB}$ under Precautionary Loss Function

The risk function of the estimator $\hat{\lambda}_{LB}, \hat{\lambda}_{SB}, \hat{\lambda}_{PB}$ under Precautionary loss function is given as

$$R_P\left(\hat{\lambda}_{PB},\lambda\right)=\lambda\left[\frac{n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}-2+\frac{\sqrt{(n\alpha-1)(n\alpha-2)}}{(n\alpha-1)}\right]$$

$$R_P\left(\hat{\lambda}_{SB},\lambda\right)=\lambda\left[\frac{n\alpha}{n\alpha-1}-1\right]$$

$$R_P\left(\hat{\lambda}_{LB},\lambda\right)=\lambda\left[\frac{n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}{a}-2+\frac{a}{(n\alpha-1)\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}\right]$$

The risk efficiencies are given as:

$$RE_P\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)=\frac{R_P\left(\hat{\lambda}_{LB},\lambda\right)}{R_P\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\left[\frac{n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}{a}-2+\frac{a}{(n\alpha-1)\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}\right]}{\left[\frac{n\alpha}{n\alpha-1}-1\right]} \quad (21)$$

$$RE_P\left(\hat{\lambda}_{LB},\hat{\lambda}_{PB}\right)=\frac{R_P\left(\hat{\lambda}_{LB},\lambda\right)}{R_P\left(\hat{\lambda}_{PB},\lambda\right)}=\frac{\left[\frac{n\alpha\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}{a}-2+\frac{a}{(n\alpha-1)\left[1-\exp\left(\frac{-a}{n\alpha+1}\right)\right]}\right]}{\left[\frac{n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}-2+\frac{\sqrt{(n\alpha-1)(n\alpha-2)}}{(n\alpha-1)}\right]} \quad (22)$$

$$RE_P\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)=\frac{R_P\left(\hat{\lambda}_{PB},\lambda\right)}{R_P\left(\hat{\lambda}_{SB},\lambda\right)}=\frac{\left[\frac{n\alpha}{\sqrt{(n\alpha-1)(n\alpha-2)}}-2+\frac{\sqrt{(n\alpha-1)(n\alpha-2)}}{(n\alpha-1)}\right]}{\left[\frac{n\alpha}{n\alpha-1}-1\right]} \quad (23)$$

5. RESULTS AND DISCUSSIONS

In order to assess the statistical performance of Bayes estimates and Mle, a simulation study is conducted. The random samples are generated from (1) with the true value of $\lambda=1$ and $\alpha=0.5$ for different samples of sizes $(n=20,40,60,80,100,120,140,160)$. We use Matlab software to generate these samples. The results of the simulation study are based on 500 repetitions. Here, Bayes estimators, risk functions and risk efficiencies of Bayes estimators are computed under different loss functions for a = 0.5,1,-1.

The estimators for the parameter and the risks and risk efficiencies are averaged over the total number of repetitions. The results of the simulation study are summarized in Table 1 to Table 4. Graphs are plotted by taking risks of $\hat{\lambda}_{LB}, \hat{\lambda}_{SB}, \hat{\lambda}_{PB}$ along the x-axis and sample size along y-axis under different loss functions to see the behaviour of Bayes estimators and to find an admissible estimator under Squared error, Linex and Precautionary loss function.

Table 1. Bayes estimators and Mle of λ and the risk and risk efficiencies of these estimators with corresponding values of $\alpha = 0.5$ and $a = 0.5$.

n	$\hat{\lambda}_{SB}$	$\hat{\lambda}_{LB}$	$\hat{\lambda}_{PB}$	$\hat{\lambda}_{MLE}$	$R_S(\hat{\lambda}_{SB},\lambda)$	$R_S(\hat{\lambda}_{LB},\lambda)$	$R_S(\hat{\lambda}_{PB},\lambda)$
20	1.1133	0.8905	1.1808	0.9986	0.1358	0.0914	0.1708
40	1.0204	0.9756	1.0309	0.9805	0.0212	0.0196	0.0222
60	1.0201	0.9753	1.0307	0.9941	0.0212	0.0196	0.0222
80	1.0266	0.9815	1.0373	0.9938	0.0212	0.0196	0.0222
100	1.0256	0.9806	1.0363	0.9912	0.0212	0.0196	0.0222
120	1.0113	0.9793	1.0187	0.9978	0.0149	0.0141	0.0154
140	1.0067	0.9788	1.0131	1.0051	0.0130	0.0124	0.0133
160	1.0081	0.9832	1.0138	1.0090	0.0115	0.0110	0.0118
n	$R_L(\hat{\lambda}_{SB},\lambda)$	$R_L(\hat{\lambda}_{LB},\lambda)$	$R_L(\hat{\lambda}_{PB},\lambda)$	$R_P(\hat{\lambda}_{SB},\lambda)$	$R_P(\hat{\lambda}_{LB},\lambda)$	$R_P(\hat{\lambda}_{PB},\lambda)$	$RE_S(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$
20	0.0187	0.0112	0.0240	0.1111	0.1389	0.1213	0.6902
40	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.6907
60	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
80	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
100	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
120	0.0019	0.0018	0.0020	0.0145	0.0151	0.0147	0.9343
140	0.0016	0.0015	0.0017	0.0127	0.0131	0.0128	0.9483
160	0.0015	0.0014	0.0015	0.0112	0.0116	0.0113	0.9544
n	$RE_S(\hat{\lambda}_{LB},\hat{\lambda}_{PB})$	$RE_S(\hat{\lambda}_{PB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{PB},\hat{\lambda}_{SB})$	$RE_P(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_P(\hat{\lambda}_{LB},\hat{\lambda}_{PB})$
20	0.5536	1.2427	0.6190	0.4837	1.2743	1.2336	1.1370
40	0.5543	1.2423	0.6196	0.4845	1.2739	1.2332	1.1367
60	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
80	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
100	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
120	0.8976	1.0409	0.8861	0.8401	1.0547	1.0599	1.0455

140	0.9200	1.0308	0.9340	0.9032	1.0342	1.0373	1.0266
160	0.9299	1.0270	0.9417	0.9142	1.0300	1.0329	1.0236
n	$RE_P\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)$						
20	1.0853						
40	1.0851						
60	1.0156						
80	1.0156						
100	1.0156						
120	1.0138						
140	1.0104						
160	1.0091						

Table 2. Bayes estimators and Mle of λ and the risk and risk efficiencies of these estimators with corresponding values of $\alpha = 0.5$ and $a = 1$.

n	$\hat{\lambda}_{SB}$	$\hat{\lambda}_{LB}$	$\hat{\lambda}_{PB}$	$\hat{\lambda}_{MLE}$	$R_S\left(\hat{\lambda}_{SB},\lambda\right)$	$R_S\left(\hat{\lambda}_{LB},\lambda\right)$	$R_S\left(\hat{\lambda}_{PB},\lambda\right)$
20	1.1133	0.8905	1.1808	0.9986	0.1358	0.0914	0.1708
40	1.0204	0.9756	1.0309	0.9805	0.0212	0.0196	0.0222
60	1.0201	0.9753	1.0307	0.9941	0.0212	0.0196	0.0222
80	1.0266	0.9815	1.0373	0.9938	0.0212	0.0196	0.0222
100	1.0256	0.9806	1.0363	0.9912	0.0212	0.0196	0.0222
120	1.0113	0.9793	1.0187	0.9978	0.0149	0.0141	0.0154
140	1.0067	0.9788	1.0131	1.0051	0.0130	0.0124	0.0133
160	1.0081	0.9832	1.0138	1.0090	0.0115	0.0110	0.0118
n	$R_L\left(\hat{\lambda}_{SB},\lambda\right)$	$R_L\left(\hat{\lambda}_{LB},\lambda\right)$	$R_L\left(\hat{\lambda}_{PB},\lambda\right)$	$R_P\left(\hat{\lambda}_{SB},\lambda\right)$	$R_P\left(\hat{\lambda}_{LB},\lambda\right)$	$R_P\left(\hat{\lambda}_{PB},\lambda\right)$	$RE_S\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)$
20	0.0187	0.0112	0.0240	0.1111	0.1389	0.1213	0.6902
40	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.6907
60	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
80	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
100	0.0027	0.0024	0.0028	0.0204	0.0215	0.0207	0.9242
120	0.0019	0.0018	0.0020	0.0145	0.0151	0.0147	0.9343
140	0.0016	0.0015	0.0017	0.0127	0.0131	0.0128	0.9483

160	0.0015	0.0014	0.0015	0.0112	0.0116	0.0113	0.9544
n	$RE_S\left(\hat{\lambda}_{LB},\hat{\lambda}_{PB}\right)$	$RE_S\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)$	$RE_L\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)$	$RE_L\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)$	$RE_P\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)$	$RE_L\left(\hat{\lambda}_{LB},\hat{\lambda}_{SB}\right)$	$RE_P\left(\hat{\lambda}_{LB},\hat{\lambda}_{PB}\right)$
20	0.5536	1.2427	0.6190	0.4837	1.2743	1.2336	1.1370
40	0.5543	1.2423	0.6196	0.4845	1.2739	1.2332	1.1367
60	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
80	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
100	0.8836	1.0460	0.9036	0.8597	1.0511	1.0549	1.0387
120	0.8976	1.0409	0.8861	0.8401	1.0547	1.0599	1.0455
140	0.9200	1.0308	0.9340	0.9032	1.0342	1.0373	1.0266
160	0.9299	1.0270	0.9417	0.9142	1.0300	1.0329	1.0236
n	$RE_P\left(\hat{\lambda}_{PB},\hat{\lambda}_{SB}\right)$						
20	1.0853						
40	1.0851						
60	1.0156						
80	1.0156						
100	1.0156						
120	1.0138						
140	1.0104						
160	1.0091						

Table 3. Bayes estimators and and Mle of λ and the risk and risk efficiencies of these estimators with corresponding values of $\alpha = 0.5$ and $a = -1$.

n	$\hat{\lambda}_{SB}$	$\hat{\lambda}_{LB}$	$\hat{\lambda}_{PB}$	$\hat{\lambda}_{MLE}$	$R_S\left(\hat{\lambda}_{SB},\lambda\right)$	$R_S\left(\hat{\lambda}_{LB},\lambda\right)$	$R_S\left(\hat{\lambda}_{PB},\lambda\right)$
20	1.0097	0.9654	1.0202	0.9895	0.0212	0.0197	0.0222
40	1.0234	0.9784	1.0340	1.0029	0.0212	0.0197	0.0222
60	1.0272	0.9894	1.0360	1.0101	0.0175	0.0165	0.0182
80	1.0078	0.9707	1.0164	0.9910	0.0175	0.0165	0.0182
100	1.0213	0.9838	1.0301	1.0043	0.0175	0.0165	0.0182
120	1.0070	0.9791	1.0134	0.9944	0.0130	0.0124	0.0133
140	1.0155	0.9874	1.0220	1.0028	0.0130	0.0124	0.0133

160	1.0142	0.9862	1.0207	1.0016	0.0130	0.0124	0.0133
n	$R_L(\hat{\lambda}_{SB},\lambda)$	$R_L(\hat{\lambda}_{LB},\lambda)$	$R_L(\hat{\lambda}_{PB},\lambda)$	$R_P(\hat{\lambda}_{SB},\lambda)$	$R_P(\hat{\lambda}_{LB},\lambda)$	$R_P(\hat{\lambda}_{PB},\lambda)$	$RE_S(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$
20	0.0103	0.0099	0.0107	0.0204	0.0207	0.0207	0.9359
40	0.0103	0.0099	0.0107	0.0204	0.0207	0.0207	0.9276
60	0.0086	0.0082	0.0088	0.0169	0.0172	0.0172	0.9393
80	0.0086	0.0082	0.0088	0.0169	0.0172	0.0172	0.9393
100	0.0086	0.0082	0.0088	0.0169	0.0172	0.0172	0.9393
120	0.0064	0.0062	0.0065	0.0127	0.0128	0.0128	0.9453
140	0.0064	0.0062	0.0065	0.0127	0.0128	0.0128	0.9542
160	0.0064	0.0062	0.0065	0.0127	0.0128	0.0128	0.9542
n	$RE_S(\hat{\lambda}_{LB},\hat{\lambda}_{PB})$	$RE_S(\hat{\lambda}_{PB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{PB},\hat{\lambda}_{SB})$	$RE_P(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_L(\hat{\lambda}_{LB},\hat{\lambda}_{SB})$	$RE_P(\hat{\lambda}_{LB},\hat{\lambda}_{PB})$
20	0.8998	1.0401	0.9563	0.9239	1.0351	1.0154	1.0018
40	0.8868	1.0460	0.9559	0.9228	1.0358	1.0149	0.9994
60	0.9048	1.0382	0.9631	0.9353	1.0297	1.0124	0.9996
80	0.9048	1.0382	0.9631	0.9353	1.0297	1.0124	0.9996
100	0.9048	1.0382	0.9631	0.9353	1.0297	1.0124	0.9996
120	0.9139	1.0343	0.9668	0.9416	1.0267	1.0112	0.9996
140	0.9277	1.0285	0.9722	0.9511	1.0222	1.0093	0.9997
160	0.9277	1.0285	0.9722	0.9511	1.0222	1.0093	0.9997
n	$RE_P(\hat{\lambda}_{PB},\hat{\lambda}_{SB})$						
20	1.0135						
40	1.0156						
60	1.0129						
80	1.0129						
100	1.0129						
120	1.0116						
140	1.0096						
160	1.0096						

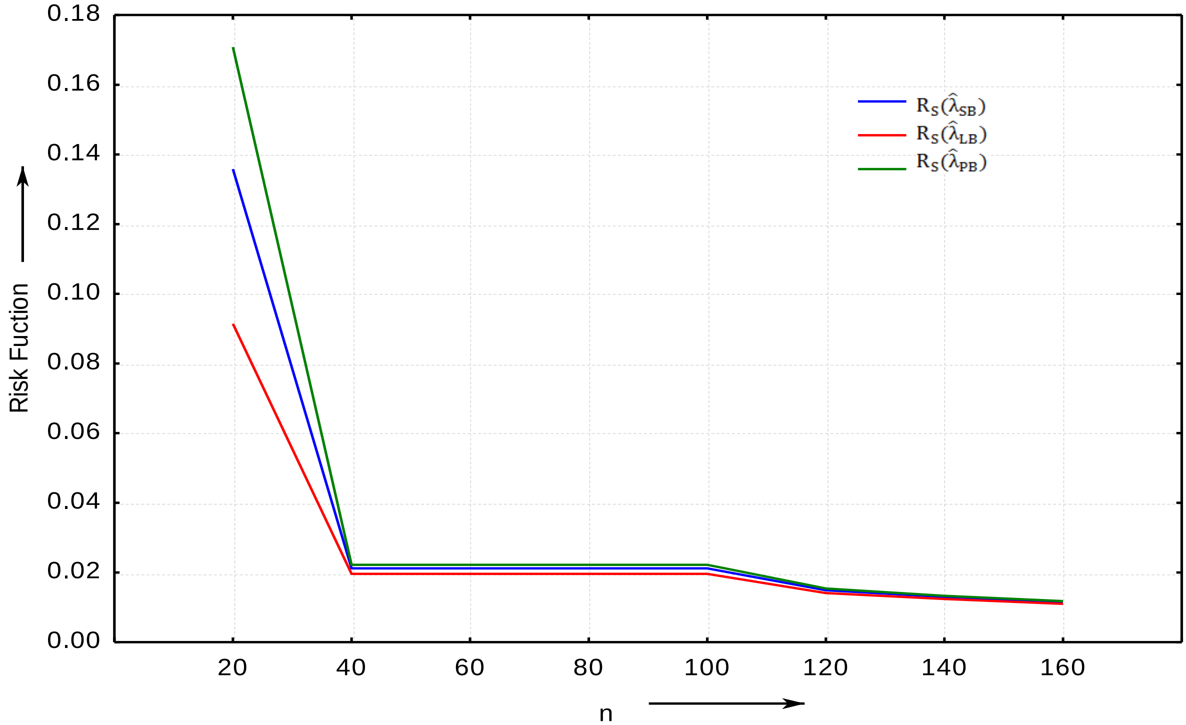

Figure 1. Graph of Risk Bayes Estimator under Square Error Loss Functions versus Sample Size for $a = 0.5$, $\alpha = 0.5$ and $\lambda = 1$.

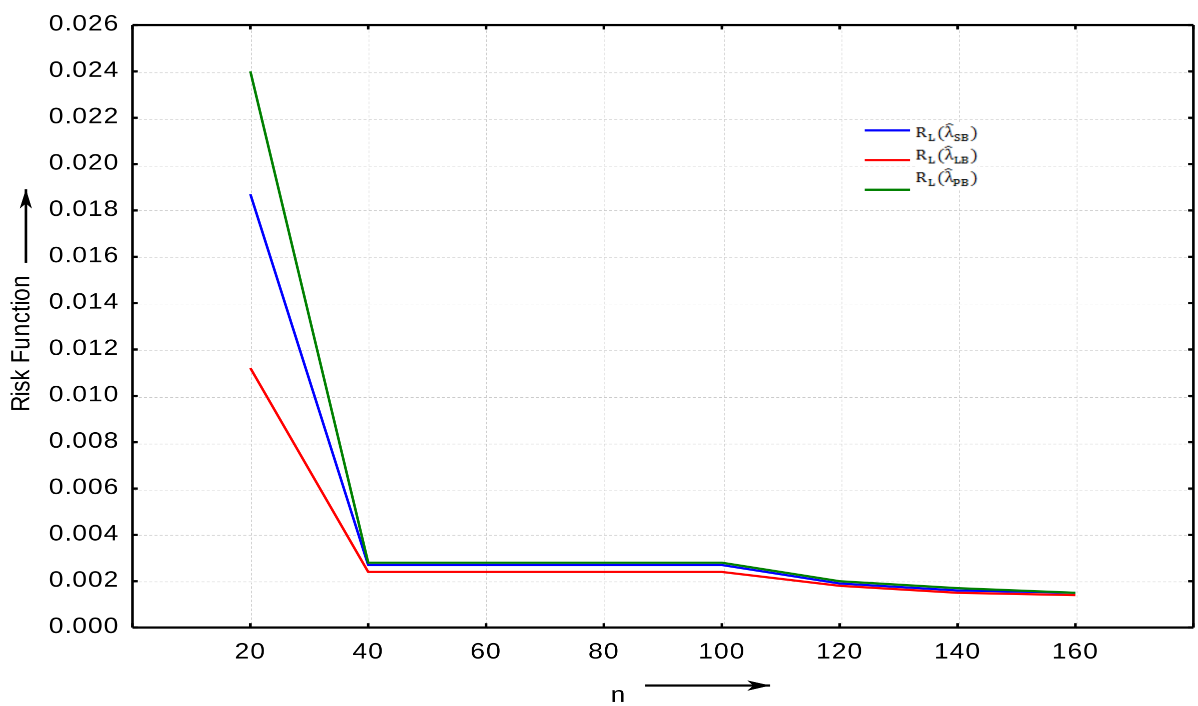

Figure 2. Graph of Risk Bayes Estimator under Linex Loss Functions versus Sample Size for $a = 0.5$, $\alpha = 0.5$ and $\lambda = 1$.

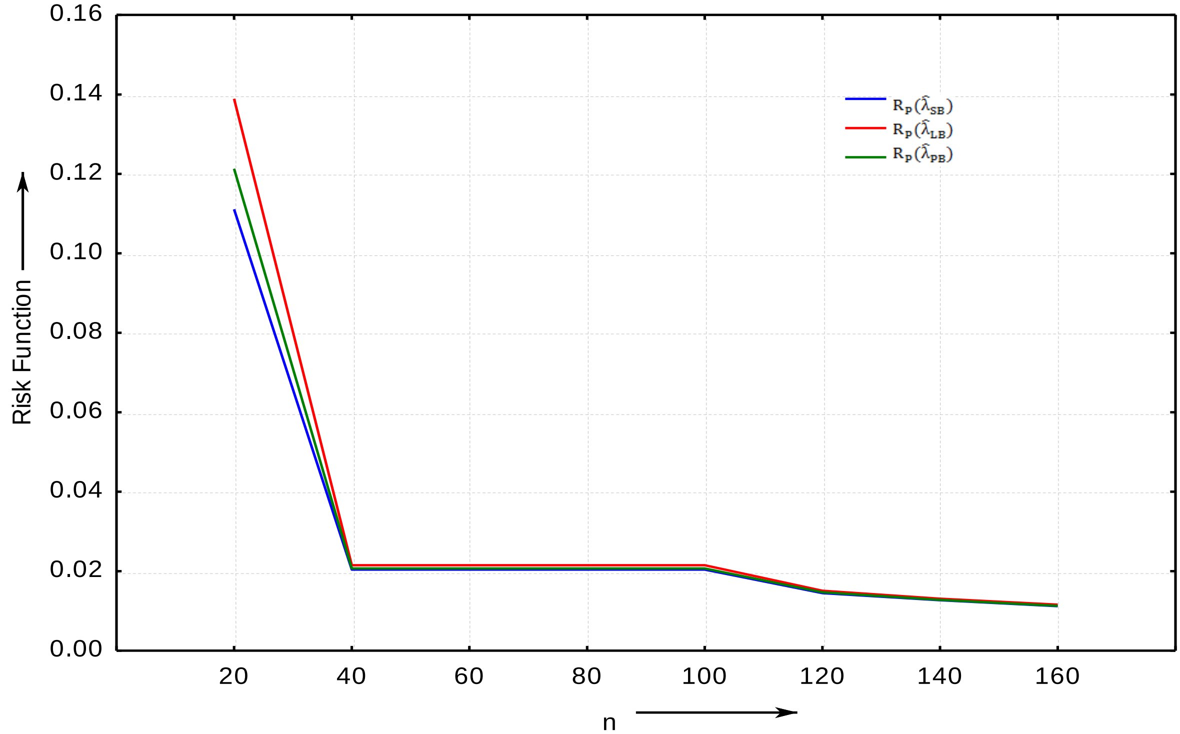

Figure 3. Graph of Risk Bayes Estimator under Precautionary Loss Functions versus Sample Size for $a = 0.5$, $\alpha = 0.5$ and $\lambda = 1$.

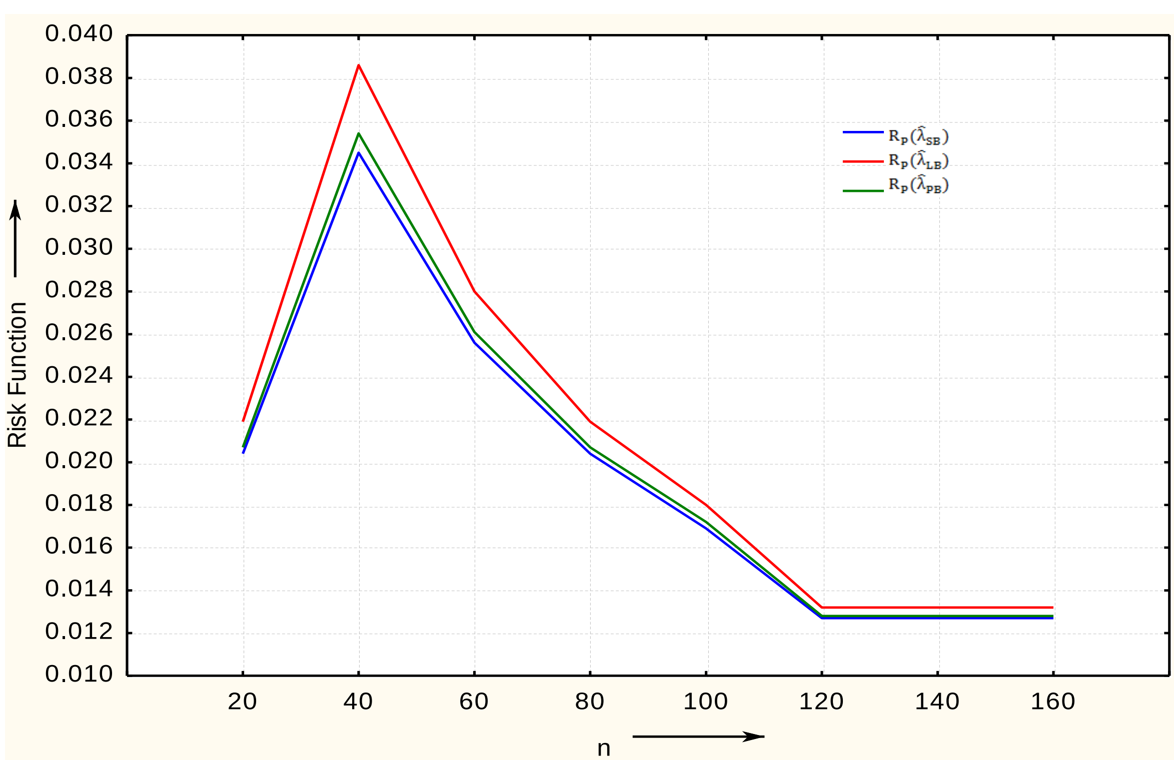

Figure 4. Graph of Risk Bayes Estimator under Precautionary Loss Functions versus Sample Size for $a = 1$, $\alpha = 0.5$ and $\lambda = 1$.

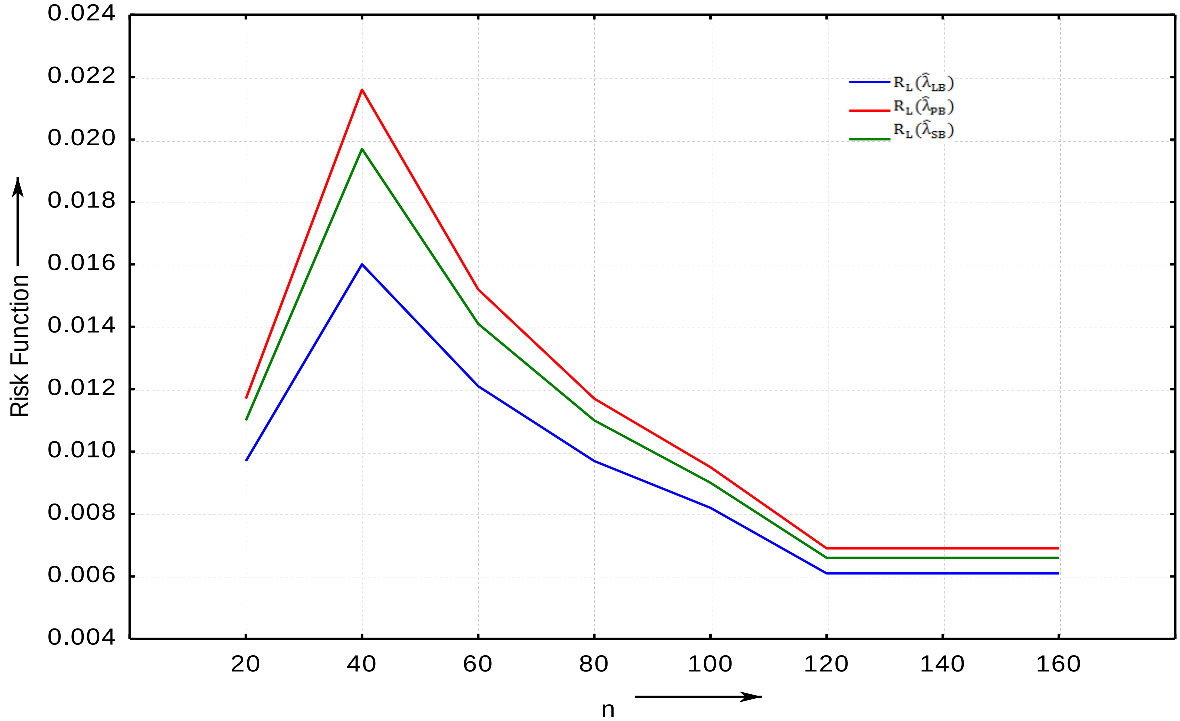

Figure 5. Graph of Risk Bayes Estimator under Linex Loss Functions versus Sample Size for $a=1$, $\alpha=0.5$ and $\lambda=1$.

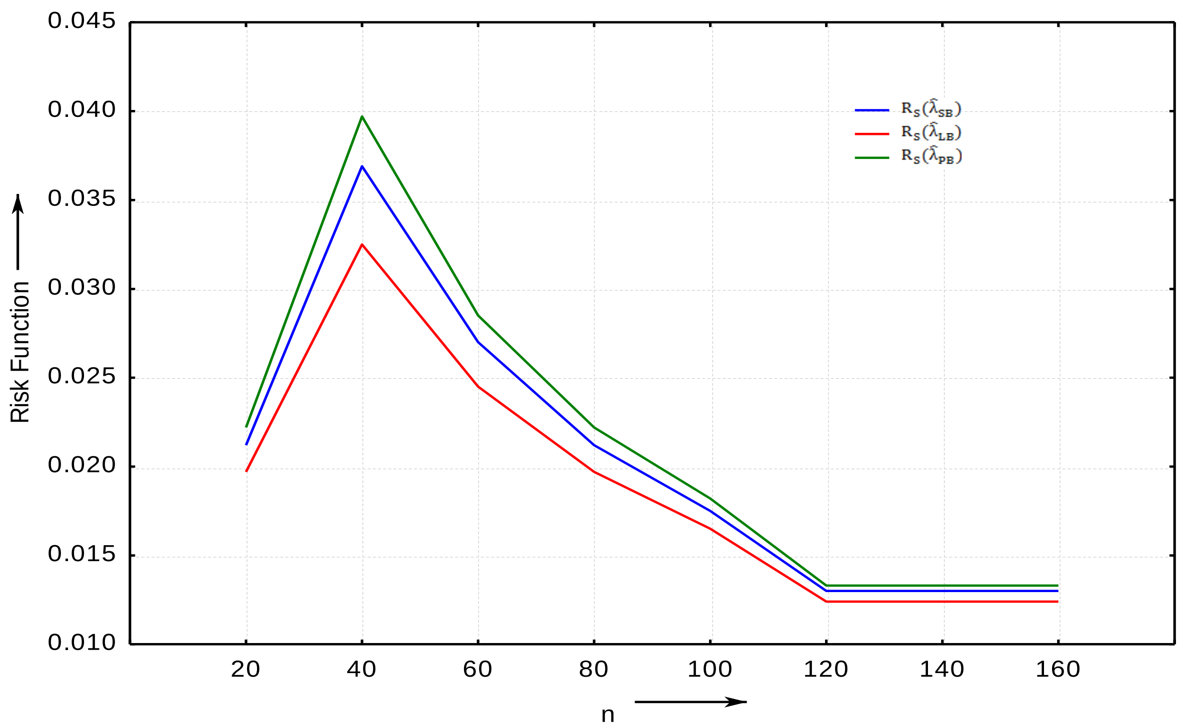

Figure 6. Graph of Risk Bayes Estimator under Square Error Loss Functions versus Sample Size for $a=1$, $\alpha=0.5$ and $\lambda=1$.

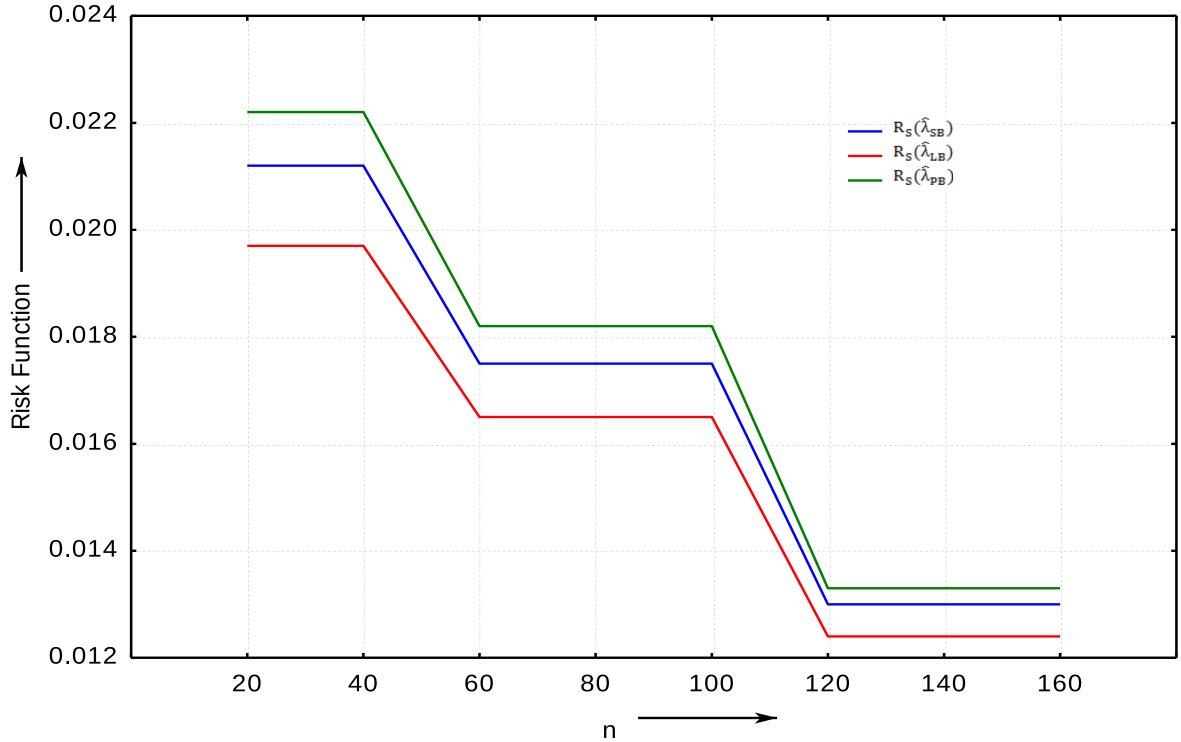

Figure 7. Graph of Risk Bayes Estimator under Square Error Loss Functions versus Sample Size for $a = -1$, $\alpha = 0.5$ and $\lambda = 1$.

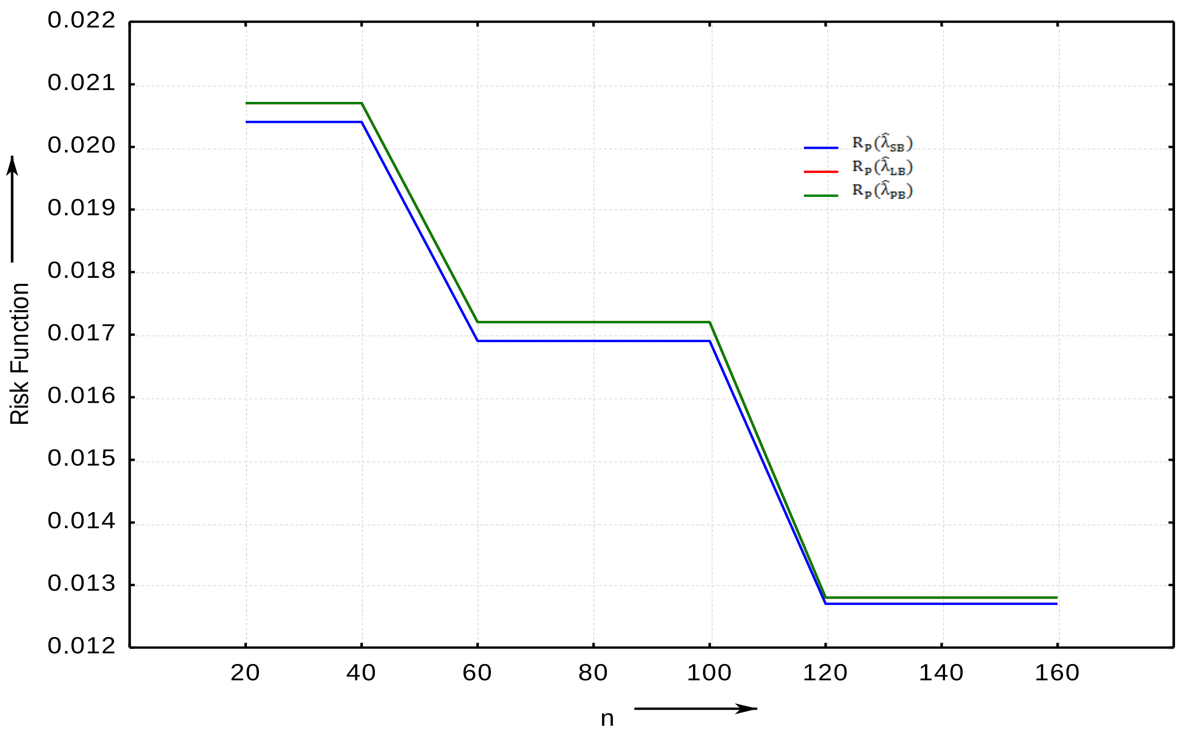

Figure 8. Graph of Risk Bayes Estimator under Precautionary Loss Functions versus Sample Size for $a = -1$, $\alpha = 0.5$ and $\lambda = 1$.

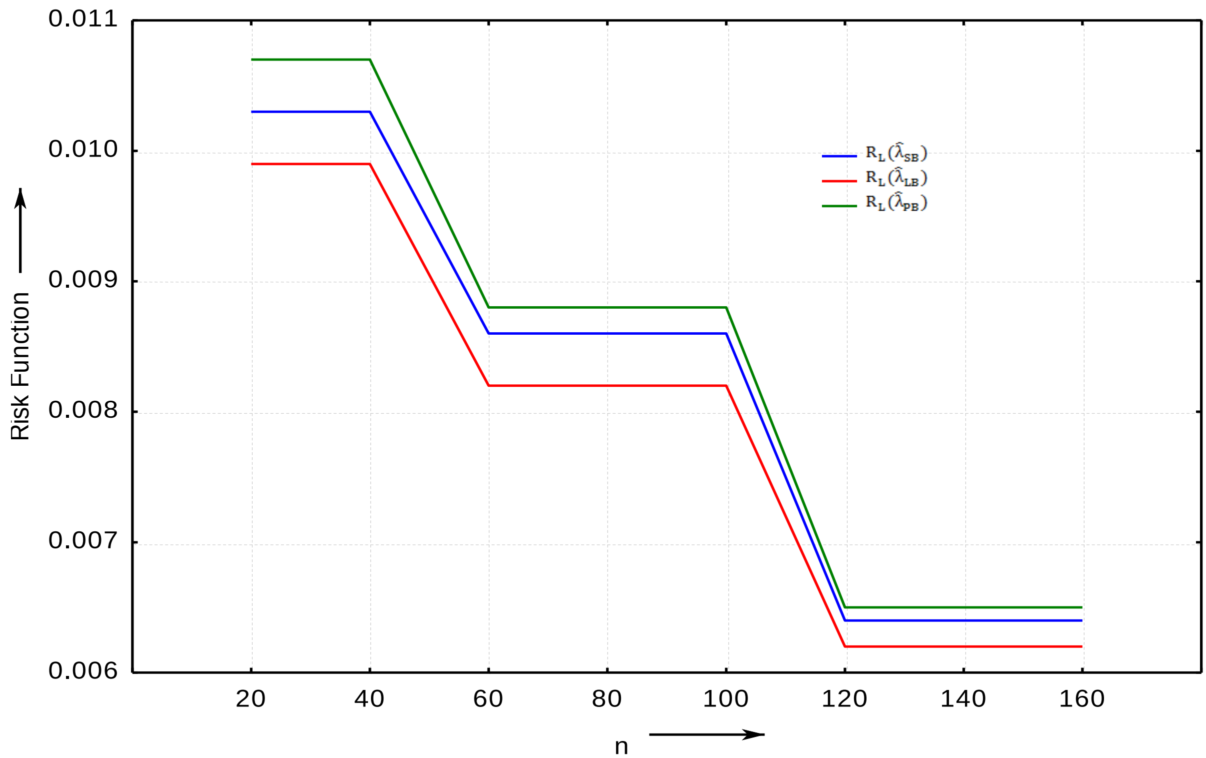

Figure 9. Graph of Risk Bayes Estimator under Linex Loss Functions versus Sample Size for $a=-1$, $\alpha=0.5$ and $\lambda=1$.

6. CONCLUSIONS

This work complements the earlier studies on the Bayesian analysis of Gamma distribution under different loss functions. From the Tables 1 to Table 3, we conclude that in situations involving estimation of Gamma scale parameter, Bayes estimator could be effectively employed than Maximum likelihood estimator as the convergence of the estimated values of Bayes estimators towards the true values tends to increase with the increase in sample size.

It is also observed that for large sample size (i.e. n = 500, 1000) Bayesian approach provides more reliable estimates. Also, Risk initially decreases as the sample size increases and then does not change much for moderate sample size and then become constant for large sample size. It indicates that the estimators are consistent.

For different values of a $\hat{\lambda}_{SB}$ is admissible under Precautionary loss function and $\hat{\lambda}_{LB}$ is admissible (has the minimum risk) under Linex and Squared error loss function. This unique feature of the proposed approach is made it more useful over the other existing approaches.

So, we suggest to use Bayes approach under Linex loss function for estimating the scale parameter of Gamma distribution.

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