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SHORT COMMUNICATION

On eigenvectors associated to a multiple eigenvalue

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ABSTRACT

If all roots of the characteristic polynomial of a matrix have distinct values, then the Faddeev-Sominsky's algorithm gives the corresponding eigenvectors. Here we exhibit the method of Gower to construct the proper vectors associated to a multiple eigenvalue.

***Keywords*:** Faddeev-Sominsky's technique, Eigenvalue problem, Gower's method

1. INTRODUCTION

Here we consider the eigenvalue problem [1]:

$$\mathbf{A}_{nxn}\,\vec{u}_k = \lambda_k\,\vec{u}_k\,, \tag{1}$$

where: the proper values λ_k are solutions of the characteristic polynomial of $\mathbf{A}$:

$$p(s) = s^n + a_1\,s^{n-1} + \cdots + a_n\,, \qquad p(\lambda_k) = 0, \tag{2}$$

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whose coefficients a_j are determined via the Leverrier-Takeno's method [2-5]:

$$r\, a_r + s_1\, a_{r-1} + s_2\, a_{r-2} + \ \dots\ + s_{r-1}\, a_1 + s_r = 0, \qquad (3)$$
$$a_0 = 1, \quad s_r = trace\ \mathbf{A}^r, \quad r = 1, 2, \dots, n,$$

that is:

$$a_1 = -s_1\,, \qquad 2!\ a_2 = (s_1)^2 - s_2\,, \qquad 3!\ a_3 = -(s_1)^3 + 3\, s_1\, s_2 - 2\, s_3\,, \qquad (4)$$
$$4!\ a_4 = (s_1)^4 - 6\,(s_1)^2\, s_2 + 8\, s_1\, s_3 + 3\,(s_2)^2 - 6\, s_4\,, \quad etc.$$

in particular, det **A** = $(-1)^n\, a_n$.

On the other hand, we have the Faddeev-Sominsky's algorithm [6-11]:

$$\mathbf{B}_1 = \mathbf{A}, \qquad q_1 = \text{tr}\,\mathbf{B}_1, \qquad \mathbf{C}_1 = \mathbf{B}_1 - q_1\,\mathbf{I},$$
$$\mathbf{B}_2 = \mathbf{C}_1\,\mathbf{A}, \qquad q_2 = \frac{1}{2}\,\text{tr}\,\mathbf{B}_2, \qquad \mathbf{C}_2 = \mathbf{B}_2 - q_2\,\mathbf{I},$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad (5)$$
$$\mathbf{B}_{n-1} = \mathbf{C}_{n-2}\,\mathbf{A}, \qquad q_{n-1} = \frac{1}{\mathrm{n}-1}\,\text{tr}\,\mathbf{B}_{n-1}, \qquad \mathbf{C}_{n-1} = \mathbf{B}_{n-1} - q_{n-1}\,\mathbf{I},$$
$$\mathbf{B}_n = \mathbf{C}_{n-1}\,\mathbf{A}, \qquad q_n = \frac{1}{\mathrm{n}}\,\text{tr}\,\mathbf{B}_n,$$

such that $q_j = -a_j$.

The Cayley-Hamilton-Frobenius theorem [1] expresses that **A** satisfies its characteristic polynomial (2):

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}. \qquad (6)$$

If $a_n \neq 0$, then **A** is non-singular and from (5) and (6) we obtain its inverse matrix:

$$\mathbf{A}^{-1} = -\frac{1}{a_n}\left(\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \cdots + a_{n-1}\mathbf{I}\right) = \frac{1}{q_n}\,\mathbf{C}_{n-1}; \qquad (7)$$

if **A** is singular the corresponding adjoint matrix [1] is given by [7]:

$$Adj\ \mathbf{A} = (-1)^{n+1}\,\mathbf{C}_{n-1}\,. \qquad (8)$$

From (5) we have the Faddeev's matrices [12]:

$$\mathbf{C}_k = \mathbf{A}^k + a_1\mathbf{A}^{k-1} + a_2\mathbf{A}^{k-2} + \ \dots\ + a_{k-1}\mathbf{A} + a_k\mathbf{I}\,, \qquad k = 1, 2, \dots, n-1, \qquad (9)$$

thus (6) and (9) imply the identity $\mathbf{C}_n = \mathbf{B}_n - q_n \mathbf{I} = \mathbf{0}$.

If the roots of (2) have distinct values, then the Faddeev-Sominsky's process allows deduce the corresponding proper vectors of **A** because for a given value of k, each column of:

$$\mathbf{Q}_k \equiv \lambda_k^{n-1}\,\mathbf{I} + \lambda_k^{n-2}\,\mathbf{C}_1 + \cdots + \mathbf{C}_{n-1}, \qquad k = 1, \dots, n\,, \tag{10}$$

verifies (1), that is:

$$\mathbf{A}\,\mathbf{Q}_k = \lambda_k\,\mathbf{Q}_k\,, \tag{11}$$

hence $\mathbf{Q}_k \neq \mathbf{0}\ \forall\, k$ and all columns of $\mathbf{Q}_k$ are proportional to each other, therefore $rank\ \mathbf{Q}_k = 1$ [13]. In fact, from [12, 14-16]:

$$\mathbf{Q}(s) \equiv s^{n-1}\,\mathbf{I} + s^{n-2}\,\mathbf{C}_1 + s^{n-3}\,\mathbf{C}_2 + \cdots + \mathbf{C}_{n-1} = \frac{p(s)}{s\,\mathbf{I} - \mathbf{A}}, \tag{12}$$

with $\mathbf{Q}(\lambda_k) = \mathbf{Q}_k$, which permits obtain (11).

Now we apply (12) to an eigenvector of **A**:

$$\mathbf{Q}(s)\,\vec{u}_k = \frac{p(s)}{s - \lambda_k}\,\vec{u}_k\,, \tag{13}$$

therefore $\mathbf{Q}_k\,\vec{u}_k = \vec{0},\ j \neq k$ when the roots of $p(s)$ have distinct values, and:

$$\mathbf{Q}_k\,\vec{u}_k = \left[\, lim_{\,s \to \lambda_k} \frac{p(s)}{s - \lambda_k} \right] \vec{u}_k = \left[\, \prod_{j=1, j\neq k}^{n} (\lambda_k - \lambda_j) \,\right] \vec{u}_k\,, \tag{14}$$

thus $\mathbf{Q}_k$ is non-null and $\vec{u}_k$ also is its eigenvector. We note that the Leverrier-Takeno-Faddeev-Sominsky algorithm was rediscovered and improved by Souriau [17] and Frame [18], see [19, 20].

However, if λ_k has multiplicity then (14) gives $\mathbf{Q}_k \vec{u}_k = \vec{0}$, that is, $\mathbf{Q}_k$ may be null without providing information on the corresponding eigenvectors. In Sec. 2 we present the Gower's method [13] to determine proper vectors associated to a multiple eigenvalue.

2. GOWER'S PROCESS

We accept that λ_k is a multiple proper value of **A**, whose minimal polynomial [1] is given by:

$$b(s) = s^m + b_1\,s^{m-1} + b_2\,s^{m-2} + \cdots + b_m\,, \qquad m \leq n, \qquad b_0 = 1, \tag{15}$$

then the columns of the matrix [13]:

$$\mathbf{E}_k = \begin{cases} \mathbf{M}_{m-2} + b_{m-1}\,\mathbf{I}, & \lambda_k = 0, \\ \lambda_k^{m-1}\,\mathbf{M}_0 + \lambda_k^{m-2}\,\mathbf{M}_1 + \cdots + \mathbf{M}_{m-1}, & \lambda_k \neq 0, \end{cases} \tag{16}$$

provide some or all the eigenvectors corresponding to λ_k , where:

$$\mathbf{M}_0 = \mathbf{A}, \qquad \mathbf{M}_j = \left(\mathbf{M}_{j-1} + b_j\,\mathbf{I}\right)\mathbf{A}, \qquad j = 1, \dots, m. \tag{17}$$

If we employ (17) into (16) we obtain the relations:

$$\mathbf{E}_k = \begin{cases} \sum_{q=0}^{m-1} b_{m-1-q}\,\mathbf{A}^q, & \lambda_k = 0, \\ \sum_{q=0}^{m-1} r_{m-1-q}\,\mathbf{A}^q, & \lambda_k \neq 0, \end{cases} \quad , \tag{18}$$

such that:

$$r_j = \sum_{t=1}^{j+1} b_{j+1-t}\,\lambda_k^t\,, \qquad j = 0, \dots, m-2, \qquad r_{m-1} = -b_m\,, \tag{19}$$

for example, if $\lambda_k = 0$:

$$\mathbf{E}_k = \begin{cases} \mathbf{A} + b_1\,\mathbf{I}, & m = 2, \\ \mathbf{A}^2 + b_1\,\mathbf{A} + b_2\,\mathbf{I}, & m = 3, \\ \mathbf{A}^3 + b_1\,\mathbf{A}^2 + b_2\,\mathbf{A} + b_3\,\mathbf{I}, & m = 4, \end{cases} \tag{20}$$

and for $\lambda_k \neq 0$:

$$\begin{gathered} m = 2: \quad \mathbf{E}_k = \lambda_k\,\mathbf{A} - b_2\,\mathbf{I}\,; \qquad m = 3: \quad \mathbf{E}_k = \lambda_k\,\mathbf{A}\,[\,\mathbf{A} + (\lambda_k + b_1)\,\mathbf{I}\,] - b_3\mathbf{I}\,; \\ m = 4: \quad \mathbf{E}_k = \lambda_k\,\mathbf{A}\,[\,\mathbf{A}^2 + (\lambda_k + b_1)\,\mathbf{A} + (\lambda_k^2 + \lambda_k\,b_1 + b_2)\,\mathbf{I}\,] - b_4\,\mathbf{I}\,, \dots \end{gathered} \tag{21}$$

We emphasize that $\mathbf{E}_k$ may not span the space of all independent proper vectors associated to λ_k , then it is appropriate to complement this approach of Gower with the technique of differentiation [19-21], in fact, if λ_k has multiplicity n_k , then the columns of the derivative $\mathbf{Q}^{(n_k-1)}(\lambda_k)$ give information about the eigenvectors $\vec{u}_r$ corresponding to this multiple proper value because:

$$\mathbf{Q}^{(j)}(\lambda_k)\,\vec{u}_r = \vec{0}, \qquad j = 0, 1, \dots, n_k - 2\,; \tag{22}$$

this property (22) is immediate from (13).

3. CONCLUSION

In this Gower's procedure is important the minimal polynomial (15), and it can be constructed via the method of minimized iterations introduced by Lanczos [22-25], with the technique of Gelbaum [26], or using the approach of Bialas-Bialas [27].

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