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Character analysis on linear elementary algebra with max-plus operation

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ABSTRACT

This paper discusses the further characteristic that exists in max-plus algebra by comparing the characteristics that exist in ordinary linear algebra. The steps taken are to examine what happens to linear algebra, then applied to max-plus algebra with respect to the existing operations. Characteristics of matrix operations and differences that occur in linear algebra and are met in max-plus algebra with max and plus operations. However, max-plus algebra does not satisfy the vector space characteristics.

***Keywords*:** Max-Plus Algebra, matrix, vector space, eigenvalues and eigenvectors

1. INTRODUCTION

Max-Plus Algebra is one part of basic science in the field of mathematics, especially algebra. Max-plus algebra is able to decipher a particular type, from a nonlinear system in linear algebra to a linear system in max-plus algebra [3]. Algebra Max-Plus was first

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introduced by Baccelli, Cohen, Olsder, and Quadrat, in 1992, and continues to be developed to date. The max-plus algebra is the set $\mathbb{R} \cup \{-\infty\}$ with R the set of all real numbers fitted with max and plus operations [11,14,19]. The basic operation of max-plus algebra is max (maximum) denoted by the symbol $\oplus$ and plus (plus) is denoted by the symbol $\otimes$, with the two operations for each $x, y \in \mathbb{R}_{max}$ [1,11,28]. Multiplication of two matrices in max-plus algebra has a similarity to the multiplication of two matrices in ordinary algebra. Multiplication of two matrices in ordinary algebra using operations $\times$ (multiplication) and $+$ (plus), while multiplication of two matrices in max-plus algebra, using (plus) and (maximum) operations respectively instead of operations $\otimes$ with $\times$ and $\oplus$ with + [23,25,27,29]. In max-plus algebra, single exponential matrix values show that for single max-plus values it gives a sequence of magnitude approaches of single values of independent classical matrix parameters [12].

Max-plus operation has been widely practiced by previous researchers. For example Imaev and Judd [8] determine eigenvector computation on an inverted matrix in max-plus algebra. Butkovic and Maccaig [4] determine the eigenvectors of integers and eigenvectors in max-plus algebra. Lu et.al [20] analyzed the reachability of timed automata using max-plus algebra. Model and algebraically analyze network problems, such as scheduling problems in a project can be seen in [2]. Krivulin [18] discusses the dynamics model of the fork-join queue network with the buffer capacity of the finite into a max-plus algebra matrix equation. The determination of eigenvalues and eigenvectors has a role in optimization solutions such as project management and scheduling can be seen in [9].

The object in the analysis is to compare the characters in the regular Linear Algebra with the max-plus Algebra characters by observing the existing operations. The goal is to get the character difference found on a regular Linear Algebra with Algebra max-plus.

2. MATERIALS AND METHODS

2. 1. Max-plus Algebra

Definition 1: *Given the set $\mathbb{R} \cup \{-\infty\}$ denoted by $\mathbb{R}_\varepsilon$, for R the set of all real numbers with maximum operation, denoted by $\oplus$ and the addition operation denoted by $\otimes$. if given any number of x and y, then x $\oplus$ y is the maximum value of one of these numbers, and x $\otimes$ y is the sum of the two numbers. Further $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is denoted by $\mathbb{R}_{max}$ and $-\infty$ denoted by ε. The ε element is a neutral element to the operation $\oplus$ and $0 = e$ is the element of identity against operation $\otimes$, and is defined:*

$$x \oplus y = max\,(x, y),$$

$$x \otimes y = x + y$$

For each $x, y \in \mathbb{R}_{max}$ [4]

2. 2. Matrix and Vector on Max-plus Algebra

Matrices and vectors have a close connection in algebra, and then in this section will be discussed about the matrices and vectors in $\mathbb{R}_{max}$.

2. 2. 1. Matrix

The $n \times m$ matrix set for $n \in \mathbb{N}$ in $\mathbb{R}_{max}$ is denoted by $\mathbb{R}_{max}^{n\times m}$. In the matrix, n denotes the number of rows and m denotes the number of columns. In general, the matrix $A \in \mathbb{R}_{max}^{n\times m}$ is written as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1m} \\ a_{21} & a_{22} & \cdots a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nm} \end{bmatrix}$$

Matrix A for the input value of i row and j column denoted by A_{ij}. The sum and maximum on the matrix and vector Max-Plus Algebra is defined in a different way maximum $\oplus$ and plus $\otimes$.

Definition 2:

a. *For $A, B \in \mathbb{R}_{max}^{nxn}$ the maximum is defined $A \oplus B$ by:*

$$[A \oplus B]_{ij} = A_{ij} \oplus B_{ij} = max(A_{ij}, B_{ij})$$

b. *The transpose of the matrix is denoted by A^T and specifically in Max-plus Algebra is defined $[A^T]_{ij} = [A]_{ji}$*

c. *Max-plus Algebra's identity matrix $n \times n$ En is defined as follows:*

$$[E_n]_{ij} = \begin{cases} 0 & if \ \ i = j \\ \varepsilon & if \ \ i \neq j \end{cases}$$

d. *For the square matrix and k positive integers, the power k on A is denoted by $A^{\otimes k}$ is defined:*

$$A^{\otimes k} = A \otimes A \otimes \cdots \otimes A \ \ For \ \ k = 0, \ \ A^{\otimes 0} = E_n$$

e. *For any matrix $A \in \mathbb{R}_{max}^{nxm}$ and any scalar $a \in \mathbb{R}_{max}$, $\alpha \otimes A$ is defined as follows:*

$$[a \otimes A]_{ij} = a \otimes [A]_{ij} \ \ [8].$$

Operations $\oplus$ and $\otimes$ in $\mathbb{R}_{max}$ can be extended to matrix operations $\mathbb{R}_{max}^{mxn}$ as in the following definitions:

Definition 3: *Given* $\mathbb{R}_{max}^{mxn} = \{A = (A_{ij}) | A_{ij} \in \mathbb{R}_{max} \ for \ i = 1,2,\ldots m \ \ and \ j = 1,2,\ldots,n\}$

a. For $a \in \mathbb{R}_{max}, A, B, \in \mathbb{R}_{max}^{mxn}$, defined $a \otimes A$ is a matrix whose elements are *ij:*

$(a \otimes A)_{ij} = a \otimes A_{ij}$ For *i = 1, 2, ... ,m* and *j = 1,2, ..., n* then $A \oplus B$ is a matrix whose elements are *ij:*$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ For *i = 1,2, .. , m* and *j = 1, 2, ... , n*

b. For $A \in \mathbb{R}_{max}^{mxp}, B \in \mathbb{R}_{max}^{pxn}$, defined $A \otimes B$ is a matrix whose elements are *ij:*

$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} A_{ik} \otimes B_{kj}$ for *i = 1,2, ... ,m* and *j = 1,2, ..., n* [22,27].

2. 2. 2. Vector

A member of $x \in \mathbb{R}_{max}^{n}$ is called a Max-Plus vector if component j of vector x is denoted by x_j or $[x]_j$. The column to j of the identity matrix E_n is known as the column vector to j in $\mathbb{R}_{max}^{n}$. This vector is denoted by $E_j = (\varepsilon, \varepsilon, \varepsilon, \cdots, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \cdots, \varepsilon)$. In other words, ε is the enter to j in the vector [8,26].

2. 3. Eigenvalues and Eigenvectors on Max-Plus Algebra

Definition 4*: Given the matrixA* $\in \mathbb{R}_{max}^{n\,x\,n}$*, then the scalar* $\lambda \in \mathbb{R}_{max}$ *is called the eigenvalue of the matrix A if there is a vector* $x \in \mathbb{R}_{max}^{n}$ *containing at least one element instead of* ε *such that*

$$A \otimes x = \lambda \otimes x$$

Furthermore, x is called the eigenvector of the matrix A [1,5,6,11].

3. RESULT AND DISCUSSION

In this section, we analyze the characteristics of Max-plus Algebra. The discussion includes: vector space, matrix characteristics and lifts characteristics in Max-plus Algebra. This discussion begins with the following vector space.

3. 1. Vector Space

In ordinary linear algebra, a vector satisfies the vector space R^n if it satisfies the ten most important properties of the vectors in R^n. based on the ten properties present in linear algebra will be proved whether the max-plus algebra also satisfies the vector space based on the ten most important properties of the vectors in $\mathbb{R}_{\varepsilon}^{n}$, taking into account the operations present in max-plus algebra.

If the following axioms are satisfied by all u, v, w on $\mathbb{R}_{\varepsilon}^{n}$ and by all scalars (number $\mathbb{R}_{\varepsilon}$) k and l, then $\mathbb{R}_{\varepsilon}^{n}$ is named as vector space

a. If u and v are objects in $\mathbb{R}_{\varepsilon}$, then $u \oplus v$ is in $\mathbb{R}_{\varepsilon}^{n}$
b. $u \oplus v = v \oplus u$
c. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$
d. There is a $\mathcal{E}$ in $\mathbb{R}_{\varepsilon}^{n}$ so $u \oplus \varepsilon = \varepsilon \oplus u = u$ for all u in $\mathbb{R}_{\varepsilon}^{n}$
e. For every u in $\mathbb{R}_{\varepsilon}^{n}$, there is $-u$ in $\mathbb{R}_{\varepsilon}^{n}$ which call negative u so $u \oplus (-u) = (-u) \oplus u = \varepsilon$

f. If k is any scalar and u is any object in $\mathbb{R}_\varepsilon^n$, then $k \otimes u$ in $\mathbb{R}_\varepsilon^n$.
g. $k \otimes (u \oplus v) = (k \otimes u) \oplus (k \otimes v)$
h. $(l \oplus k) \otimes u = (l \otimes u) \oplus (k \otimes u)$
i. $l \otimes (k \otimes u) = (l \otimes k) \otimes u$
j. $e \otimes u = u$

Evidence:

Take $u = (u_1, u_2, \cdots, u_n)$; $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots, w_n)$ $\forall\, u, v, w \in \mathbb{R}_\varepsilon^n$

a.
$$\begin{aligned} u \oplus v &= (u_1, u_2, \cdots, u_n) \oplus (v_1, v_2, \cdots, v_n) \\ &= (u_1 \oplus v_1, u_2 \oplus v_2, \cdots, u_n \oplus v_n) \\ &= \big(max\,(u_1, v_1),\ max(u_2, v_2),\ \cdots\ ,\ max(u_n, v_n)\big) \quad \in \mathbb{R}_\varepsilon^n \end{aligned}$$

b.
$$\begin{aligned} u \oplus v &= (u_1, u_2, \cdots, u_n) \oplus (v_1, v_2, \cdots, v_n) \\ &= (u_1 \oplus v_1, u_2 \oplus v_2, \cdots, u_n \oplus v_n) \\ &= \big(max\,(u_1, v_1),\ max(u_2, v_2),\ \cdots,\ max(u_n, v_n)\big) \\ &= \big(max\,(v_1, u_1),\ max(v_2, u_2),\ \cdots,\ max(v_n, u_n)\big) \\ &= (v_1 \oplus u_1, v_2 \oplus u_2, \cdots, v_n \oplus u_n) \\ &= (v_1, v_2, \cdots, v_n) \oplus (u_1, u_2, \cdots, u_n) \\ &= v \oplus u \end{aligned}$$

c.
$$\begin{aligned} u \oplus (v \oplus w) &= (u_1, u_2, \cdots, u_n) \oplus \big((v_1, v_2, \cdots, v_n) \oplus (w_1, w_2, \cdots, w_n)\big) \\ &= (u_1, u_2, \cdots, u_n) \oplus (v_1 \oplus w_1, v_2 \oplus w_2, \cdots, v_n \oplus w_n) \\ &= \big(u_1 \oplus (v_1 \oplus w_1), u_2 \oplus (v_2 \oplus w_2), \cdots, u_n \oplus (v_n \oplus w_n)\big) \\ &= \Big(max\big(u_1, max\,(v_1, w_1)\big),\ max\big(u_2, max(v_2, w_2)\big), \\ &\quad \cdots,\ max\big(u_n, max(v_n, w_n)\big)\Big) \\ &= \big(max\,(u_1, v_1, w_1),\ max(u_2, v_2, w_2),\ \cdots,\ max(u_n, v_n, w_n)\big) \\ &= \big(max(max\,(u_1, v_1), w_1),\ max(max(u_2, v_2), w_2),\ \cdots, \\ &\quad max\,(max(u_n, v_n), w_n)\big) \\ &= \big((u_1, u_2, \cdots, u_n) \oplus (v_1, v_2, \cdots, v_n)\big) \oplus (w_1, w_2, \cdots, w_n) \\ &= (u \oplus v) \oplus w \end{aligned}$$

d. $\exists\ \varepsilon = (\varepsilon, \varepsilon, \cdots, \varepsilon) \in \mathbb{R}_\varepsilon^n$ so

$$u \oplus \varepsilon = \varepsilon \oplus u = u$$

$$\begin{aligned} \varepsilon \oplus u &= (\varepsilon, \varepsilon, \cdots, \varepsilon) \oplus (u_1, u_2, \cdots, u_n) \\ &= (\varepsilon \oplus u_1, \varepsilon \oplus u_2, \cdots, \varepsilon \oplus u_n) \\ &= \big(max(\varepsilon, u_1), max(\varepsilon, u_2), \cdots, max(\varepsilon, u_n)\big) \\ &= (u_1, u_2, \cdots, u_n) \\ &= u \end{aligned}$$

e. $\forall\ u \in \mathbb{R}_\varepsilon^n \quad \exists\ -u \in \mathbb{R}_\varepsilon^n$ so

$u \oplus (-u) = \varepsilon$

In this section cannot be satisfied $u \oplus (-u) = \varepsilon$ because the max operation of its inverse element does not exist, as seen in the following theorem.

Theorem 1. *If the operation* $\oplus$ *in* $\mathbb{R}_{max}$ $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ *is idempotent, then the inverse element to the operation* $\oplus$ *does not exist.*

Contradictions with $x \neq \varepsilon$. so the inverse element to the operation $\oplus$ does not exist.

f. Let k scalar for $k \in \mathbb{R}_\varepsilon$ then

$$\begin{aligned} k \otimes u &= k \otimes (u_1, u_2, \cdots, u_n) \\ &= (k \otimes u_1, k \otimes u_2, \cdots, k \otimes u_n) \\ &= (k + u_1, k + u_2, \cdots, k + u_n) \quad \in \mathbb{R}_\varepsilon^n \end{aligned}$$

g.
$$\begin{aligned} k \otimes (u \oplus v) &= k \otimes \big((u_1, u_2, \cdots, u_n) \oplus (v_1, v_2, \cdots, v_n)\big) \\ &= k \otimes (u_1 \oplus v_1, u_2 \oplus v_2, \cdots, u_n \oplus v_n) \\ &= \big(k \otimes (u_1 \oplus v_1), k \otimes (u_2, \oplus v_2), \cdots, k \otimes (u_n \oplus v_n)\big) \\ &= \big(k + max(u_1, v_1),\ k + max(u_2, v_2), \cdots, k + max(u_n, v_n)\big) \\ &= \big(max(k + u_1, k + v_1),\ max(k + u_2, k + v_2), \cdots, max(k + u_n, k + v_n)\big) \\ &= \big((k \otimes u_1) \oplus (k \otimes v_1), (k \otimes u_2) \oplus (k \otimes v_2), \cdots, (k \otimes u_n) \oplus (k \otimes v_n)\big) \\ &= \big(k \otimes (u_1, u_2, \cdots, u_n)\big) \oplus (k \otimes (v_1, v_2, \cdots, v_n)) \\ &= (k \otimes u) \oplus (k \otimes v) \end{aligned}$$

h.
$$\begin{aligned} (l \oplus k) \otimes u &= (l \oplus k) \otimes (u_1, u_2, \cdots, u_n) \\ &= \big((l \oplus k) \otimes u_1, (l \oplus k) \otimes u_2, \cdots, (l \oplus k) \otimes u_n\big) \\ &= (max(l, k) + u_1, max(l, k) + u_2, \cdots, max(l, k) + u_n) \\ &= \big(max(l + u_1, k + u_1), max(l + u_2, k + u_2), \cdots, max(l + u_n, k + u_n)\big) \\ &= \big(l \otimes (u_1, u_2, \cdots, u_n)\big) \oplus \big(k \otimes (u_1, u_2, \cdots, u_n)\big) \\ &= (l \otimes u) \oplus (k \otimes u) \end{aligned}$$

i.
$$\begin{aligned} l \otimes (k \otimes u) &= l \otimes \big(k \otimes (u_1, u_2, \cdots, u_n)\big) \\ &= \big(l \otimes (k \otimes u_1), l \otimes (k \otimes u_2), \cdots, l \otimes (k \otimes u_n)\big) \\ &= \big(l + (k + u_1), l + (k + u_2), \cdots, l + (k + u_n)\big) \\ &= \big((l + k) + u_1, (l + k) + u_2, \cdots, (l + k) + u_n\big) \\ &= \big((l \otimes k) \otimes u_1, (l \otimes k) \otimes u_2, \cdots, (l \otimes k) \otimes u_n\big) \\ &= (l \otimes k) \otimes (u_1, u_2, \cdots, u_n) \\ &= (l \otimes k) \otimes u \end{aligned}$$

j.
$$\begin{aligned} e \otimes u &= e \otimes (u_1, u_2, \cdots, u_n) \\ &= (e \otimes u_1, e \otimes u_2, \cdots, e \otimes u_n) \\ &= (0 + u_1, 0 + u_2, \cdots, 0 + u_n) \\ &= (u_1, u_2, \cdots, u_n) = u \end{aligned}$$

From the above proof, it appears that $\mathbb{R}_{\varepsilon}^{n}$ does not meet the properties of the vector space. Because at max operation, it has no inverse element so $u \oplus (-u) = (-u) \oplus u = \varepsilon$ can not be fulfilled. In this case $\mathbb{R}_{\varepsilon}^{n}$ is not a vector space.

3. 2. Characteristics Matrix Max-Plus Algebra

In linear algebra, the operation of the matrix is known to some of the properties found in the operation of the matrix. As with linear algebra, max-plus algebra also has the same characteristics in linear algebra with respect to the operations present in max-plus algebra. The rules of arithmetic that exist in linear algebra such as commutative law, associative law and distributive law also apply to max-plus algebra. For more details can be seen in the following discussion:

Some of the following properties apply to any matrix A, B, and C with corresponding sizes and undefined matrix operations, for loyal scalar $\alpha, \beta \in \mathbb{R}_{max}$

a. $A \oplus B = B \oplus A$

b. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

c. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

d. $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$

e. $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$

f. $(\alpha \oplus \beta) \otimes A = (\alpha \otimes A) \oplus (\beta \otimes A)$

g. $\alpha \otimes (A \oplus B) = (\alpha \otimes A) \oplus (\alpha \otimes B)$

h. $\alpha \otimes (\beta \otimes A) = (\alpha \otimes \beta) \otimes A$

Evidence:

a. Take any matrix $A, B \in \mathbb{R}_{max}^{m\,x\,n}$ so

$$A \oplus B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} \oplus b_{11}) & (a_{12} \oplus b_{12}) & \cdots & (a_{1n} \oplus b_{1n}) \\ (a_{21} \oplus b_{21}) & (a_{22} \oplus b_{22}) & \cdots & (a_{2n} \oplus b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1} \oplus b_{m1}) & (a_{m2} \oplus b_{m2}) & \cdots & (a_{mn} \oplus b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} max(a_{11}, b_{11}) & max(a_{12}, b_{12}) & \cdots & max(a_{1n}, b_{1n}) \\ max(a_{21}, b_{21}) & max(a_{22}, b_{22}) & \cdots & max(a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(a_{m1}, b_{m1}) & max(a_{m2}, b_{m2}) & \cdots & max(a_{mn}, b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} max(b_{11}, a_{11}) & max(b_{12}, a_{12}) & \cdots & max(b_{1n}, a_{1n}) \\ max(b_{21}, a_{21}) & max(b_{22}, a_{22}) & \cdots & max(b_{2n}, a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(b_{m1}, a_{m1}) & max(b_{m2}, a_{m2}) & \cdots & max(b_{mn}, a_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} (b_{11} \oplus a_{11}) & (b_{12} \oplus a_{12}) & \cdots & (b_{1n} \oplus a_{1n}) \\ (b_{21} \oplus a_{21}) & (b_{22} \oplus a_{22}) & \cdots & (b_{2n} \oplus a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (b_{m1} \oplus a_{m1}) & (b_{m2} \oplus a_{m2}) & \cdots & (b_{mn} \oplus a_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \oplus \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= B \oplus A$$

b. Take any matrix $A, B, C \in \mathbb{R}_{max}^{m\,x\,n}$ so

$$(A \oplus B) \oplus C = \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \right) \oplus \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} \oplus b_{11}) \oplus c_{11} & (a_{12} \oplus b_{12}) \oplus c_{12} & \cdots & (a_{1n} \oplus b_{1n}) \oplus c_{1n} \\ (a_{21} \oplus b_{21}) \oplus c_{21} & (a_{22} \oplus b_{22}) \oplus c_{22} & \cdots & (a_{2n} \oplus b_{2n}) \oplus c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1} \oplus b_{m1}) \oplus c_{m1} & (a_{m2} \oplus b_{m2}) \oplus c_{m2} & \cdots & (a_{mn} \oplus b_{mn}) \oplus c_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} max(max(a_{11}, b_{11}), c_{11}) & max(max(a_{12}, b_{12}), c_{12}) & \cdots & max(max(a_{1n}, b_{1n}), c_{1n}) \\ max(max(a_{21}, b_{21}), c_{21}) & max(max\,(a_{22}, b_{22}), c_{22}) & \cdots & max(max(a_{2n}, b_{2n}), c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(max(a_{m1}, b_{m1}), c_{m1}) & max(max(a_{m2}, b_{m2}), c_{m2}) & \cdots & max(max(a_{mn}, b_{mn}), c_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} max(a_{11}, b_{11}, c_{11}) & max(a_{12}, b_{12}, c_{12}) & \cdots & max(a_{1n}, b_{1n}, c_{1n}) \\ max(a_{21}, b_{21}, c_{21}) & max(a_{22}, b_{22}, c_{22}) & \cdots & max(a_{2n}, b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(a_{m1}, b_{m1}, c_{m1}) & max(a_{m2}, b_{m2}, c_{m2}) & \cdots & max(a_{mn}, b_{mn}, c_{mn}) \end{bmatrix} =$$

$$\begin{bmatrix} max\left(a_{11}, max(b_{11}, c_{11})\right) & max\left(a_{12}, max(b_{12}, c_{12})\right) & \cdots & max\left(a_{1n}, max(b_{1n}, c_{1n})\right) \\ max\left(a_{21}, max(b_{21}, c_{21})\right) & max\left(a_{22}, max(b_{22}, c_{22})\right) & \cdots & max\left(a_{2n}, max(b_{2n}, c_{2n})\right) \\ \vdots & \vdots & \ddots & \vdots \\ max\left(a_{m1}, max(b_{m1}, c_{m1})\right) & max\left(a_{m2}, max(b_{m2}, c_{m2})\right) & \cdots & max\left(a_{mn}, max(b_{mn}, c_{mn})\right) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \oplus (b_{11} \oplus c_{11}) & a_{12} \oplus (b_{12} \oplus c_{12}) & \cdots & a_{1n} \oplus (b_{1n} \oplus c_{1n}) \\ a_{21} \oplus (b_{21} \oplus c_{21}) & a_{22} \oplus (b_{22} \oplus c_{22}) & \cdots & a_{2n} \oplus (b_{2n} \oplus c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \oplus (b_{m1} \oplus c_{m1}) & a_{m2} \oplus (b_{m2} \oplus c_{m2}) & \cdots & a_{mn} \oplus (b_{mn} \oplus c_{mn}) \end{bmatrix}$$

$$= (A \oplus B) \oplus C$$

c. Take any matrix $A \in \mathbb{R}_{max}^{m\,x\,p}$, $B \in \mathbb{R}_{max}^{p\,x\,q}$, and $C \in \mathbb{R}_{max}^{q\,x\,m}$. The row element i column j matrix (A ⊗ B) ⊗ C is as follows:

$$\begin{aligned}[(A \otimes B) \otimes C]_{ij} &= \oplus_{k=1}^{q} \left(\oplus_{l=1}^{p} A_{i,l} \otimes B_{l,k}\right) \otimes C_{k,j} \\ &= \oplus_{k=1}^{q} \oplus_{l=1}^{p} A_{i,l} \otimes B_{l,k} \otimes C_{k,j} \\ &= \oplus_{l=1}^{p} A_{i,l} \otimes \left(\oplus_{k=1}^{q} B_{l,k} \otimes C_{k,j}\right) \\ &= [A \otimes (B \otimes C)]_{\boldsymbol{ij}} \text{ For } i \in n \text{ and } j \in m.\end{aligned}$$

d. Take any matrix $A \in \mathbb{R}_{max}^{n\,x\,q}$ and $B,\ C \in \mathbb{R}_{max}^{q\,x\,m}$. The row element i column j matrix (A ⊗ B) ⊗ C is as follows:

$$\begin{aligned}[(A \otimes (B \oplus C))]_{ij} &= \oplus_{k=1}^{p} A_{i,k} \otimes \left(B_{k,j} \oplus C_{k,j}\right) \\ &= \oplus_{k=1}^{p} \left(A_{i,k} \otimes B_{k,j} \oplus A_{i,k} \otimes C_{k,j}\right) \\ &= \left(\oplus_{k=1}^{p} A_{i,k} \otimes B_{k,j}\right) \oplus \left(\oplus_{k=1}^{p} A_{i,k} \otimes C_{k,j}\right) \\ &= [(A \otimes B)]_{ij} \oplus [(A \otimes C)]_{ij},\end{aligned}$$

e. Take any matrix $A, B, C \in \mathbb{R}_{max}^{n\,x\,n}$

$$\begin{aligned}(A \oplus B) \otimes C &= \otimes_{k=1}^{n} (a_{ik} \oplus b_{ik}) \otimes c_{kj} \\ &= \otimes_{k=1}^{n} \left(a_{ik} \otimes c_{kj} \oplus b_{ik} \otimes c_{kj}\right) \\ &= \left(\otimes_{k=1}^{n} \left(a_{ik} \otimes c_{kj}\right)\right) \oplus \left(\otimes_{k=1}^{n} \left(b_{ik} \otimes c_{kj}\right)\right) \\ &= (A \otimes C) \oplus (B \otimes C)\end{aligned}$$

f. Take any matrix $A \in \mathbb{R}_{max}^{m\,x\,n}$

$$(\alpha \oplus \beta) \otimes A = (\alpha \oplus \beta) \otimes \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha \oplus \beta) \otimes a_{11} & (\alpha \oplus \beta) \otimes a_{12} & \cdots & (\alpha \oplus \beta) \otimes a_{1n} \\ (\alpha \oplus \beta) \otimes a_{21} & (\alpha \oplus \beta) \otimes a_{22} & \cdots & (\alpha \oplus \beta) \otimes a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha \oplus \beta) \otimes a_{m1} & (\alpha \oplus \beta) \otimes a_{m2} & \cdots & (\alpha \oplus \beta) \otimes a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} max(\alpha, \beta) + a_{11} & max(\alpha, \beta) + a_{12} & \cdots & max(\alpha, \beta) + a_{1n} \\ max(\alpha, \beta) + a_{21} & max(\alpha, \beta) + a_{22} & \cdots & max(\alpha, \beta) + a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ max(\alpha, \beta) + a_{m1} & max(\alpha, \beta) + a_{m2} & \cdots & max(\alpha, \beta) + a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} max(\alpha + a_{11}, \beta + a_{11}) & max(\alpha+a_{12}, \beta+a_{12}) & \cdots & max(\alpha+a_{1n}, \beta+a_{1n}) \\ max(\alpha+a_{21}, \beta+a_{21}) & max(\alpha+a_{22}, \beta+a_{22}) & \cdots & max(\alpha+a_{2n}, \beta+a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(\alpha+a_{m1}, \beta+a_{m1}) & max(\alpha + a_{m2}, \beta + a_{m2}) & \cdots & max(\alpha+a_{mn}, \beta+a_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha \otimes a_{11} & \alpha \otimes a_{12} & \cdots & \alpha \otimes a_{1n} \\ \alpha \otimes a_{21} & \alpha \otimes a_{22} & \cdots & \alpha \otimes a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \otimes a_{m1} & \alpha \otimes a_{m2} & \cdots & \alpha \otimes a_{mn} \end{bmatrix} \oplus \begin{bmatrix} \beta \otimes a_{11} & \beta \otimes a_{12} & \cdots & \beta \otimes a_{1n} \\ \beta \otimes a_{21} & \beta \otimes a_{22} & \cdots & \beta \otimes a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta \otimes a_{m1} & \beta \otimes a_{m2} & \cdots & \beta \otimes a_{mn} \end{bmatrix}$$

$$= (\alpha \otimes A) \oplus (\beta \otimes A)$$

g. Take any matrix $A, B \in \mathbb{R}_{max}^{m \times n}$ so

$$a \otimes (A \oplus B) = a \otimes \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a \otimes (a_{11} \oplus b_{11}) & a \otimes (a_{12} \oplus b_{12}) & \cdots & a \otimes (a_{1n} \oplus b_{1n}) \\ a \otimes (a_{21} \oplus b_{21}) & a \otimes (a_{22} \oplus b_{22}) & \cdots & a \otimes (a_{2n} \oplus b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ a \otimes (a_{m1} \oplus b_{m1}) & a \otimes (a_{m2} \oplus b_{m2}) & \cdots & a \otimes (a_{mn} \oplus b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} a + max(a_{11}, b_{11}) & a + max(a_{12}, b_{12}) & \cdots & a + max(a_{1n}, b_{1n}) \\ a + max(a_{21}, b_{21}) & a + max(a_{22}, b_{22}) & \cdots & a + max(a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ a + max(a_{m1}, b_{m1}) & a + max(a_{m2}, b_{m2}) & \cdots & a + max(a_{mn}, b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} max(a + a_{11}, a + b_{11}) & max(a + a_{12}, a + b_{12}) & \cdots & max(a + a_{1n}, a + b_{1n}) \\ max(a + a_{21}, a + b_{21}) & max(a + a_{22}, a + b_{22}) & \cdots & max(a + a_{2n}, a + b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ max(a + a_{m1}, a + b_{m1}) & max(a + a_{m2}, a + b_{m2}) & \cdots & max(a + a_{mn}, a + b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} a \otimes a_{11} & a \otimes a_{12} & \cdots & a \otimes a_{1n} \\ a \otimes a_{21} & a \otimes a_{22} & \cdots & a \otimes a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a \otimes a_{m1} & a \otimes a_{m2} & \cdots & a \otimes a_{mn} \end{bmatrix} \oplus \begin{bmatrix} a \otimes b_{11} & a \otimes b_{12} & \cdots & a \otimes b_{1n} \\ a \otimes b_{21} & a \otimes b_{22} & \cdots & a \otimes b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a \otimes b_{m1} & a \otimes b_{m2} & \cdots & a \otimes b_{mn} \end{bmatrix}$$

$$= (a \otimes A) \oplus (a \otimes B)$$

h. Take any matrix $A \in \mathbb{R}_{max}^{m \times n}$ so

$$a \otimes (\beta \otimes A) = a \otimes \left(\beta \otimes \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a \otimes (\beta \otimes a_{11}) & a \otimes (\beta \otimes a_{12}) & \cdots & a \otimes (\beta \otimes a_{1n}) \\ a \otimes (\beta \otimes a_{21}) & a \otimes (\beta \otimes a_{22}) & \cdots & a \otimes (\beta \otimes a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ a \otimes (\beta \otimes a_{m1}) & a \otimes (\beta \otimes a_{m2}) & \cdots & a \otimes (\beta \otimes a_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} (a \otimes \beta) \otimes a_{11} & (a \otimes \beta) \otimes a_{12} & \cdots & (a \otimes \beta) \otimes a_{1n} \\ (a \otimes \beta) \otimes a_{21} & (a \otimes \beta) \otimes a_{22} & \cdots & (a \otimes \beta) \otimes a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (a \otimes \beta) \otimes a_{m1} & (a \otimes \beta) \otimes a_{m2} & \cdots & (a \otimes \beta) \otimes a_{mn} \end{bmatrix}$$

$$= (a \otimes \beta) \otimes \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= (a \otimes \beta) \otimes A$$

3. 3. Characteristic of appointment Max-Plus Algebra

Operating $\oplus$ and $\otimes$ in max-plus algebra, operation $\otimes$ is performed first from operation $\oplus$. Some properties in the operation of the numbers on max-plus algebra are as follows:

For each $n, m \in \mathbb{N}$ and $x, y \in \mathbb{R}_{max}$ so:

a. $x^{\otimes^m} \otimes x^{\otimes^n} = x^{\otimes^{(m \otimes n)}}$

b. $\left(x^{\otimes^m}\right)^{\otimes n} = x^{\otimes^{(m^{\otimes n})}}$

c. $x^{\otimes^1} = x$

d. $x^{\otimes^m} \otimes y^{\otimes^m} = (x \otimes y)^{\otimes^m}$

Evidence:

a. $$x^{\otimes^m} \otimes x^{\otimes^n} = (\underbrace{x \otimes x \otimes x \cdots \otimes x}_{m}) + (\underbrace{x \otimes x \otimes x \cdots \otimes x}_{n})$$

$$= (\underbrace{x + x + x \cdots + x}_{m}) + (\underbrace{x + x + x \cdots + x}_{n})$$

$$= m.x + n.x = (m + n)x$$

$$= (m \otimes n)x$$

$$= x^{\otimes^{(m \otimes n)}}$$

b. $$\left(x^{\otimes^m}\right)^{\otimes n} = (\underbrace{x \otimes x \otimes x \cdots \otimes x}_{m})^{\otimes n} = (\underbrace{x + x + x \cdots + x}_{m})^{\otimes n}$$

$$= (m.x)^{\otimes n} = (\underbrace{m.x \otimes m.x \otimes m.x \cdots \otimes m.x}_{n})$$

$$= (\underbrace{m.x + m.x + m.x \cdots + m.x}_{n}) = n.(m.x) = (n.m.x)$$

$$= x^{\otimes (m^{\otimes n})}$$

c. $x^{\otimes 1} = 1.x = x$

d. $$x^{\otimes m} \otimes y^{\otimes m} = (\underbrace{x \otimes x \otimes x \cdots \otimes x}_{m}) + (\underbrace{y \otimes y \otimes y \cdots \otimes y}_{m})$$

$$= (\underbrace{x + x + x \cdots + x}_{m}) + (\underbrace{y + y + y \cdots + y}_{m})$$

$$= m.x + m.y = m(x + y) = m(x \otimes y)$$

$$= (x \otimes y)^{\otimes m}$$

4. CONCLUSION

Based on the results of the research it can be concluded that the matrix of max-plus algebra has the same characteristics as the matrix in ordinary algebra with respect to the operations present in max-plus algebra. The properties of vector space on max-plus algebra do not satisfy the properties of vector spaces in ordinary linear algebra

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