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SHORT COMMUNICATION

On Stirling numbers

V. Barrera-Figueroa[1], J. López-Bonilla[2,*], R. López-Vázquez[2], S. Vidal-Beltrán[2]

[1] SEPI-UPIITA, Instituto Politécnico Nacional, Posgrado en Tecnología Avanzada, Av. IPN 2580, Col. Barrio la Laguna 07340, CDMX, México

[2] ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, Col. Lindavista 07738, CDMX, México

*E-mail address: jlopezb@ipn.mx

ABSTRACT

We exhibit an approach for the recursive formula obtained by Cereceda for the sums of powers of integers. Besides, we show that the definitions of Janjic for the Stirling numbers are consequences of the formulas of Hoppe and Quaintance-Gould.

***Keywords*:** Generalized chain rule of differentiation, Stirling and Bernoulli numbers, Hoppe's formula, Sums of powers of integers

1. INTRODUCTION

Here we consider the sums of powers of integers:

$$S_k(n) \equiv 1^k + 2^k + \cdots + n^k, \qquad k \geq 0, \tag{1}$$

and we show the identity:

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$$\sum_{m=1}^{n-1} S_{k-1}(m) = \sum_{q=2}^{n} (q-1)! \binom{n+1}{q+1} S_{k-1}^{[q-1]}, \quad n \geq 1, \quad k \geq 2, \tag{2}$$

where: $S_j^{[r]}$ are the Stirling numbers of the second kind [1-3].

In Sec. 2 we give a simple proof of the property [4]:

$$n\, S_{k-1}(n) - S_k(n) = \sum_{m=1}^{n-1} S_{k-1}(m), \qquad k \geq 2, \quad n \geq 1. \tag{3}$$

Finally, we prove that (2) and (3) imply the Cereceda's recursive formula [4]:

$$(k+1)\, S_k(n) = k\left(n + \tfrac{1}{2}\right) S_{k-1}(n) - \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r}\, S_r(n), \quad n \geq 1, \quad k \geq 2, \tag{4}$$

where: B_j are the Bernoulli numbers [1, 5-7].

Janjic [8] employs the relations:

$$\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} S_n^{[k]}\, u^k \frac{d^k}{du^k} f(u), \qquad u = e^x, \tag{5}$$

$$\frac{d^n}{dx^n} f(v) = \frac{1}{x^n} \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k} f(v), \qquad v = \ln x, \tag{6}$$

where: $f(t)$ is an arbitrary function, to define the Stirling numbers $S_n^{[k]}$ and $S_n^{(k)}$ [1].

In Sec. 3 we show that (5) and (6) are consequences of the formulas of Hoppe [9, 10] and Quaintance-Gould [1], respectively.

2. CERECEDA'S IDENTITY

We know the relation [11, 12]:

$$S_{k-1}(m) = \sum_{l=1}^{m} l! \binom{m+1}{l+1} S_{k-1}^{[l]}, \tag{7}$$

and the expression [1]:

$$\sum_{k=j}^{N} \binom{k}{j} = \binom{N+1}{j+1}, \tag{8}$$

then:

$$\sum_{m=1}^{n-1} S_{k-1}(m) \overset{(7)}{=} \sum_{l=1}^{n-1} l!\, S_{k-1}^{[l]} \sum_{m=l}^{n-1} \binom{m+1}{l+1} \overset{(8)}{=} \sum_{l=1}^{n-1} l! \binom{n+1}{l+2} S_{k-1}^{[l]},$$

which is equivalent to (2), q.e.d.

On the other hand, from (1):

$$n\, S_{k-1}(n) - S_k(n) = n\,(1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + (n-1)^{k-1} + n^{k-1}) -$$

$$-(1^k + 2^k + 3^k + \cdots + (n-1)^k + n^k) = (n-1)1^{k-1} + (n-2)2^{k-1} + \cdots + \big(n-(n-1)\big)(n-1)^{k-1},$$

$$= 1^{k-1} + (1^{k-1} + 2^{k-1}) + (1^{k-1} + 2^{k-1} + 3^{k-1}) + \cdots + (1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + (n-1)^{k-1}),$$

thus (3) is proved.

Now we study the following relation involving Bernoulli numbers:

$$A \equiv \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r}\, S_r(n) = Q - \binom{k}{k-1} B_1\, S_{k-1}(n) - B_0\, S_k(n) = Q + \frac{k}{2}\, S_{k-1}(n) - S_k(n), \tag{9}$$

where:

$$Q \equiv \sum_{r=1}^{k} \binom{k}{r} B_{k-r}\, S_r(n). \tag{10}$$

We have the properties [1]:

$$\binom{k}{r} B_{k-r} = k \sum_{q=r}^{k} \frac{1}{q}\, S_{k-1}^{[q-1]}\, S_q^{(r)}\,, \tag{11}$$

$$\sum_{r=j}^{q} S_q^{(r)}\, S_r^{[j]} = \delta_{jq}\,, \tag{12}$$

where: $S_q^{(r)}$ are the Stirling numbers of the first kind [7], hence from (7), (10) and (11):

$$Q = k \sum_{q=1}^{k} \frac{1}{q}\, S_{k-1}^{[q-1]} \sum_{r=1}^{q} \sum_{j=1}^{r} j! \binom{n+1}{j+1} S_q^{(r)}\, S_r^{[j]},$$

$$= k \sum_{q=1}^{k} \frac{1}{q}\, S_{k-1}^{[q-1]} \sum_{j=1}^{q} j! \binom{n+1}{j+1} \sum_{r=j}^{q} S_q^{(r)}\, S_r^{[j]} \overset{(12)}{=} k \sum_{q=1}^{k} (q-1)! \binom{n+1}{q+1} S_{k-1}^{[q-1]}\,. \tag{13}$$

Finally, if we employ (2), (3) and (13) into (9) we obtain the Cereceda's recursive formula indicated in (4), q.e.d.

3. JANJIC'S RELATIONS

In the Hoppe's expression [9, 10]:

$$\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{d^k}{du^k} f(u) \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{k-j} \frac{d^n}{dx^n} u^j, \tag{14}$$

we apply $u = e^x$, thus $\frac{d^n}{dx^n} u^j = j^n u^j$, then (5) is immediate by the Euler's result [1]:

$$S_n^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n . \tag{15}$$

We know the property [1]:

$$n! \binom{y}{n} = \sum_{k=0}^{n} S_n^{(k)} y^k, \tag{16}$$

where: we can use $y = \frac{d}{dv}$ to deduce:

$$n! \binom{\frac{d}{dv}}{n} = \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k}, \tag{17}$$

besides, we have the relation [1]:

$$\frac{d^n}{dx^n} f(v) = \frac{n!}{x^n} \binom{\frac{d}{dv}}{n} f(v), \qquad v = \ln x, \tag{18}$$

therefore (17) and (18) imply (6).

The expressions (5) and (6) are the formulas (8.13) and (12.34) in [1], respectively.

4. CONCLUSIONS

We show that the relations obtained by Cereceda [4] and Janjic [8] can be deduced from known expressions for the Bernoulli and Stirling numbers, and the generalized chain rule of differentiation.

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