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SHORT COMMUNICATION

Brans-Edgar-Ludwig equations for empty spacetimes

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ABSTRACT

We use the Newman-Penrose formalism to deduce the compatibility conditions for the Bianchi identities associated to vacuum spacetimes.

***Keywords*:** Petrov types, Bianchi identities, Null tetrad formalism, Debever-Penrose principal directions, Brans-Edgar-Ludwig equations

1. INTRODUCTION

The Bianchi identities are very important in General Relativity, because they allow study the evolution of gravity from one region to another of the spacetime. In particular, the Bianchi identities ensure that the gravitational field equations determine the dynamics of the geometry of spacetime without determining also the coordinate system. Due to the fact that these relations are of differential nature, they require the fulfillment of certain integrability conditions to admit solution. In the literature, such conditions have been obtained by means of

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the so-called Newman-Penrose (NP) formalism [1-4] for empty R_4. Ludwig [5] was one of the first researchers to deduce these important relations, although without performing a detailed analysis of the resulting equations for the different Petrov types. On the other hand, Brans [6] (with the aid of a computer) only obtained the compatibility conditions for the type I, and subsequently Edgar [7] showed that the twelve Brans relations are, in fact, easy to find without the need of numerical methods, besides of pointing out some printing mistakes in [6]. Thereby, the now so-called Brans-Edgar-Ludwig equations (BEL), constitute the integrability conditions for the Bianchi identities, which in turn, are also integrability conditions for the 18 NP equations [1-4]. In this work, we derive for completeness purposes the BEL equations for empty R_4 of the types I, II, III and N; the NP equations together with the Bianchi and BEL equations, belong to a complete integrable ensemble. Therefore, our results contribute to the search for exact solutions to the Einstein field equations.

2. POST-BIANCHI EQUATIONS

Here, we employ the notation and conventions of [3]. In the null-tetrad formalism [1-4] the linear differential operators $\delta, \bar{\delta}, \Delta, D$ are fundamental, as well as their corresponding commutators [3]:

$$
\begin{aligned}
[\Delta, D] &= (\gamma + \bar{\gamma})\, D + (\varepsilon + \bar{\varepsilon})\, \Delta - (\tau + \bar{\pi})\, \bar{\delta} - (\bar{\tau} + \pi)\, \delta, \\
[\delta, D] &= (\bar{\alpha} + \beta - \bar{\pi})\, D + \kappa\, \Delta - \sigma\, \bar{\delta} - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\, \delta, \\
[\delta, \Delta] &= -\bar{\nu}\, D + (\tau - \bar{\alpha} - \beta)\, \Delta + \bar{\lambda}\, \bar{\delta} + (\mu - \gamma + \bar{\gamma})\, \delta, \\
[\bar{\delta}, \delta] &= (\bar{\mu} - \mu)\, D + (\bar{\rho} - \rho)\, \Delta - (\bar{\alpha} - \beta)\, \bar{\delta} - (\bar{\beta} - \alpha)\, \delta,
\end{aligned}
\tag{1}
$$

in the deduction of the integrability conditions. In the procedure followed here to the attainment of the BEL equations, we employ the Bianchi identities for each Petrov type, making use of the respective commutators.

a). Empty spacetime of type III.

The NP tetrad can be chosen, without loss of generality, such that $\psi_r = 0,\ r \neq 3$ with $\psi_3 \neq 0$ [3]. Furthermore, the Goldberg-Sachs [8] and Kundt-Thompson [9] theorems imply $\kappa = \sigma = 0$. In these conditions, the Bianchi identities will be:

$$
\begin{aligned}
\bar{\delta}\psi_3 &= -2(\alpha + 2\pi)\psi_3, \quad \delta\psi_3 = 2(\tau - \beta)\psi_3, \\
\Delta\psi_3 &= -2(\gamma + 2\mu)\psi_3, \quad D\psi_3 = 2(\rho - \varepsilon)\psi_3.
\end{aligned}
\tag{2}
$$

The substitution of (2) into (1) allows obtain the integrability relations of BEL:

$$\bar{\delta}\rho + 2\,D\pi - \rho(\alpha + \bar{\beta}) + 2\pi(\varepsilon - \bar{\varepsilon}) = 0,$$

$$\bar{\delta}\tau + 2\,\delta\pi + 2\pi(\beta - \bar{\alpha}) + \tau(\bar{\beta} - \alpha) + 2\bar{\rho}\mu - \rho\bar{\mu} = 0,$$

$$\Delta\tau + 2\,\delta\mu - \bar{\nu}\rho + 2\mu(\bar{\alpha} + \beta) - 2\pi\bar{\lambda} + \tau(\bar{\gamma} - \gamma) = 0, \qquad (3)$$

$$\Delta\pi - \bar{\delta}\mu + \nu\rho - \lambda\tau + \pi(\gamma - \bar{\gamma} + \bar{\mu}) + \mu(\bar{\tau} - \bar{\beta} - \alpha) - \frac{1}{2}\psi_3 = 0,$$

for empty R_4 of the type III. It is important to note that, in the deduction of (3), the 18 NP equations [3] are not explicitly employed

b). Vacuum R_4 of type N.

For this type there exists a canonical tetrad with the property $\psi_r = 0,\ r \neq 4$ with $\psi_4 \neq 0$ [3]. On the other hand, due to the fact that $\kappa = \sigma = 0$ [8, 9], the Bianchi identities reduce to:

$$D\psi_4 = (\rho - 4\varepsilon)\psi_4, \qquad \delta\psi_4 = (\tau - 4\beta)\psi_4. \qquad (4)$$

From (4) and the corresponding commutators (1) can be obtained conditions that are already contained into the 18 NP equations, which means that for the Petrov type N, the BEL equations are not generated.

c). Empty 4-space of type II.

In this case, the null tetrad can be taken such that $\psi_0 = \psi_1 = \psi_3 = 0$ with ψ_2 and ψ_4 different from zero, furthermore $\kappa = \sigma = 0$. Consequently, the Bianchi identities turn out to be:

$$D\psi_2 = 3\rho\psi_2, \quad \Delta\psi_2 = -3\mu\psi_2, \quad \bar{\delta}\psi_2 = -3\pi\psi_2, \quad \delta\psi_2 = 3\tau\psi_2,$$

$$D\psi_4 = (\rho - 4\varepsilon)\psi_4 - 3\lambda\psi_2, \qquad \delta\psi_4 = (\tau - 4\beta)\psi_4 - 3\nu\psi_2, \qquad (5)$$

similarly to the previous cases, the application of (1) gives rise to the BEL equations:

$$\delta\mu + \Delta\tau - \rho\bar{\nu} - \bar{\lambda}\pi + \mu(\bar{\alpha} + \beta) + \tau(\bar{\gamma} - \gamma) = 0,$$

$$\Delta\rho + D\mu + \mu(\varepsilon + \bar{\varepsilon}) - \rho(\gamma + \bar{\gamma}) + \tau\bar{\tau} - \pi\bar{\pi} = 0,$$

$$\bar{\delta}\rho + D\pi - \rho(\alpha + \bar{\beta}) - \pi(\bar{\varepsilon} - \varepsilon) = 0, \qquad (6)$$

$$\Delta\pi - \bar{\delta}\mu + \rho\nu - \lambda\tau - \mu(\alpha + \bar{\beta} - \bar{\tau}) - \pi(\bar{\gamma} - \gamma - \bar{\mu}) = 0,$$

for empty R_4 of type II. In this case, it is worth noting that (6) is in agreement with the BEL equations for the type D [6].

d). Vacuum metric of type I.

For vacuum solutions of type I we use a canonical tetrad of Newman-Penrose [1, 3, 10] such that $\psi_0 = \psi_4$ and $\psi_1 = \psi_3 = 0$, hence the Bianchi identities [11] adopt the form:

$$\begin{aligned} &\bar{\delta}\psi_0 = (4\alpha - \pi)\psi_0 + 3\kappa\psi_2\,, && \delta\psi_0 = (\tau - 4\beta)\psi_0 - 3\nu\psi_2\,, \\ &\Delta\psi_0 = (4\gamma - \mu)\psi_0 + 3\sigma\psi_2\,, && D\psi_0 = (\rho - 4\varepsilon)\psi_0 - 3\lambda\psi_2\,, \\ &\bar{\delta}\psi_2 = \kappa\psi_0 - 3\pi\psi_2\,, && \delta\psi_2 = -\nu\psi_0 + 3\tau\psi_2\,, \\ &\Delta\psi_2 = \sigma\psi_0 - 3\mu\psi_2\,, && D\psi_2 = -\lambda\psi_0 + 3\rho\psi_2\,, \end{aligned} \tag{7}$$

whose integrability constraints are deduced with the commutators (1), thus we obtain the BEL equations [6, 7]:

(a) $\psi_0[-\Delta\lambda - D\sigma - \lambda(2\mu + 3\gamma - \bar{\gamma}) + \sigma(2\rho + 3\varepsilon - \bar{\varepsilon}) + \kappa(\tau + \bar{\pi}) - \nu(\bar{\tau} + \pi)] +$

$+ 3\psi_2[\Delta\rho + D\mu - \rho(\gamma + \bar{\gamma}) + \mu(\varepsilon + \bar{\varepsilon}) - \pi\bar{\pi} + \tau\bar{\tau}] = 0,$

(b) $\psi_0[D\nu - \delta\lambda + \lambda(\bar{\alpha} + 5\beta + 2\tau - \bar{\pi}) + \nu(\bar{\varepsilon} - 2\rho - \bar{\rho} - 5\varepsilon)] +$

$+ 3\psi_2[\delta\rho - D\tau + \rho(\bar{\pi} - \bar{\alpha} - \beta) + \tau(\bar{\rho} + \varepsilon - \bar{\varepsilon}) + \mu\kappa - \pi\sigma] = 0,$

(c) $\psi_0\left[-D\kappa - \bar{\delta}\lambda + \kappa(3\varrho + 3\varepsilon + \bar{\varepsilon}) - \bar{\kappa}\sigma - \bar{\sigma}\nu - \lambda\left(3\pi + 3\alpha - \bar{\beta}\right)\right] +$

$+ 3\psi_2\left[\bar{\delta}\rho + D\pi - \rho\left(\alpha + \bar{\beta}\right) - \pi(\bar{\varepsilon} - \varepsilon) + \mu\bar{\kappa} + \tau\bar{\sigma}\right] = 0,$

(d) $\psi_0\left[\delta\sigma + \Delta\nu - \bar{\nu}\lambda - \bar{\lambda}\kappa + \sigma(\bar{\alpha} - 3\beta - 3\tau) + \nu(3\gamma + \bar{\gamma} + 3\mu)\right] +$

$+ 3\psi_2\left[-\delta\mu - \Delta\tau + \bar{\nu}\rho - \mu(\bar{\alpha} + \beta) + \tau(\gamma - \bar{\gamma}) + \pi\bar{\lambda}\right] = 0,$

(e) $\psi_0\left[\bar{\delta}\sigma - \Delta\kappa + \sigma\left(2\pi - \bar{\tau} + 5\alpha + \bar{\beta}\right) + \kappa(\bar{\gamma} - 5\gamma - \bar{\mu} - 2\mu)\right] +$

$+ 3\psi_2\left[\Delta\pi - \bar{\delta}\mu + \nu\rho - \mu\left(-\bar{\tau} + \alpha + \bar{\beta}\right) - \pi(\bar{\gamma} - \gamma - \bar{\mu}) - \lambda\tau\right] = 0,$

(f) $\psi_0\left[-\delta\kappa - \bar{\delta}\nu + \lambda(\bar{\mu} - \mu) + \sigma(\rho - \bar{\rho}) + \kappa(2\tau + \bar{\alpha} + 3\beta) - \nu\left(2\pi + \bar{\beta} + 3\alpha\right)\right] +$

$+ 3\psi_2\left[\bar{\delta}\tau + \delta\pi - \rho\bar{\mu} + \bar{\rho}\mu + \pi(\beta - \bar{\alpha}) + \tau\left(\bar{\beta} - \alpha\right)\right] = 0,$

(8)

(g) $\psi_0[\Delta\rho + D\mu + \mu(\varepsilon + \bar{\varepsilon}) - \rho(\gamma + \bar{\gamma}) - \pi\bar{\pi} + \tau\bar{\tau} - 4(\Delta\varepsilon + D\gamma + \gamma\bar{\varepsilon} - \bar{\gamma}\varepsilon -$

$\alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi)] + 3\psi_2[-\Delta\lambda - D\sigma - \sigma(2\rho + 5\varepsilon + \bar{\varepsilon}) + \lambda(2\mu + 5\gamma + \bar{\gamma})$

$+\kappa(\tau + \bar{\pi}) - \nu(\bar{\tau} + \pi)] = 0,$

(h) $\psi_0[\delta\rho - D\tau + \tau(\bar{\rho} + \varepsilon - \bar{\varepsilon}) + \rho(\bar{\pi} - \bar{\alpha} - \beta) + \sigma(3\pi + 8\alpha) - \kappa(3\mu + 8\gamma)] +$

$+ 3\psi_2[D\nu - \delta\lambda + \nu(2\rho - \bar{\rho} + 3\varepsilon + \bar{\varepsilon}) - \lambda(\bar{\pi} - \bar{\alpha} + 3\beta + 2\tau)] = 0,$

(i) $\psi_0[\bar{\delta}\rho + D\pi + \pi(\varepsilon - \bar{\varepsilon}) - \rho(\alpha + \bar{\beta}) + \mu\bar{\kappa} + \bar{\sigma}\tau - 4(\bar{\delta}\varepsilon + D\alpha + \gamma\bar{\kappa} + \beta\bar{\sigma} +$

$\varepsilon(\pi - \bar{\beta}) - \alpha(\rho + \bar{\varepsilon}))] + 3\psi_2[-\bar{\delta}\lambda - D\kappa + \kappa(\bar{\varepsilon} - 5\varepsilon - \rho) + \lambda(5\alpha + \bar{\beta} + \pi)$

$-\sigma\bar{\kappa} - \bar{\sigma}\nu] = 0,$

(j) $\psi_0[-\delta\mu - \Delta\tau + \tau(\gamma - \bar{\gamma}) + \bar{\nu}\rho - \mu(\bar{\alpha} + \beta) + \bar{\lambda}\pi - 4(-\delta\gamma - \Delta\beta - \beta(\mu + \bar{\gamma}) -$

$\gamma(\bar{\alpha} - \tau) + \varepsilon\bar{\nu} + \bar{\lambda}\alpha)] + 3\psi_2[\delta\sigma + \Delta\nu - \nu(5\gamma + \mu - \bar{\gamma}) + \sigma(\bar{\alpha} + 5\beta + \tau) - \bar{\nu}\lambda -$
$\bar{\lambda}\kappa] = 0,$

(k) $\psi_0[-\bar{\delta}\mu + \Delta\pi + 3(\tau\lambda - \nu\rho) + 8(\beta\lambda - \varepsilon\nu) + \pi(\gamma - \bar{\gamma} + \bar{\mu}) + \mu(\bar{\tau} - \alpha - \bar{\beta})] +$

$3\psi_2[\bar{\delta}\sigma - \Delta\kappa + \kappa(3\gamma + \bar{\gamma} + 2\mu - \bar{\mu}) + \sigma(\bar{\beta} - 3\alpha - 2\pi - \bar{\tau})] = 0,$

(l) $\psi_0[\bar{\delta}\tau + \delta\pi + \tau(\bar{\beta} - \alpha) + \pi(\beta - \bar{\alpha}) + \mu\bar{\rho} - \rho\bar{\mu} - 4(\bar{\delta}\beta + \delta\alpha + \beta\bar{\beta} - \alpha\bar{\alpha} +$

$\gamma(\bar{\rho} - \rho) + \varepsilon(\mu - \bar{\mu}))] + 3\psi_2[-\bar{\delta}\nu - \delta\kappa + \lambda(\bar{\mu} - \mu) + \sigma(\rho - \bar{\rho}) + \kappa(\bar{\alpha} - 5\beta - 2\tau) +$

$\nu(2\pi - \bar{\beta} + 5\alpha)] = 0.$

If we remember the Newman-Penrose relations [1, 3, 11], in our canonical tetrad for empty space of type I:

(a) $D\rho - \bar{\delta}\kappa = (\rho + \varepsilon + \bar{\varepsilon})\rho + \sigma\bar{\sigma} - \bar{\kappa}\tau - (3\alpha + \bar{\beta} - \pi)\kappa,$
(b) $D\sigma - \delta\kappa = (\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \psi_0,$
(c) $D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa,$
(d) $D\alpha - \bar{\delta}\varepsilon = (\varrho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi,$
(e) $D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon,$
(f) $D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \psi_2,$
(g) $D\lambda - \bar{\delta}\pi = (\rho - 3\varepsilon + \bar{\varepsilon})\lambda + (\pi + \alpha - \bar{\beta})\pi + \bar{\sigma}\mu - \nu\bar{\kappa},$
(h) $D\mu - \delta\pi = (\bar{\rho} - \varepsilon - \bar{\varepsilon})\mu + (\bar{\pi} - \bar{\alpha} + \beta)\pi + \sigma\lambda - \nu\kappa + \psi_2,$
(i) $D\nu - \Delta\pi = (\mu + \gamma - \bar{\gamma})\pi + \bar{\tau}\mu + (\bar{\pi} + \tau)\lambda - (3\varepsilon + \bar{\varepsilon})\nu,$

(9)

(j) $\Delta\lambda - \bar{\delta}\nu = (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \psi_0$,
(k) $\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa$,
(l) $\delta\alpha - \bar{\delta}\beta = (\rho + \varepsilon)\mu - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma - \varepsilon\bar{\mu} - \psi_2$,
(m) $\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\alpha + \bar{\beta} + \pi)\mu - \pi\bar{\mu} + (\bar{\alpha} - 3\beta)\lambda$,
(n) $\delta\nu - \Delta\mu = (\mu + \gamma + \bar{\gamma})\mu + \lambda\bar{\lambda} - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu$,
(o) $\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - 2\beta)\gamma + (\tau + \beta)\mu - \sigma\nu - \varepsilon\bar{\nu} + \beta\bar{\gamma} + \alpha\bar{\lambda}$,
(p) $\delta\tau - \Delta\sigma = (\mu - 3\gamma + \bar{\gamma})\sigma + \bar{\lambda}\rho + (\tau + \beta - \bar{\alpha})\tau - \kappa\bar{\nu}$,
(q) $\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - \sigma\lambda + (\bar{\beta} - \alpha - \bar{\tau})\tau + \nu\kappa - \psi_2$,
(r) $\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma$,

then we can see that into (8) only there are 9 independent equations, in fact, with (9.b, h, j, q) in (8.a) we deduce (8.f), and if we use (9.c, i, k, m) in (8.b) and (8.e) we obtain (8.k) and (8.h), respectively.

The expressions (8) represent the integrability conditions to guarantee the existence of solutions of the system (7).

3. CONCLUSIONS

In this work we have not addressed the study of empty R_4 of type D because this is a particular case of the type I contained in (9) [6, 7]. The results obtained here, equations (3) and (6), do not appear in the literature and their integrability conditions lead to 0 = 0. This results and their verification for the type I carried out by Brans, imply the theorem:

The equations of Newman-Penrose, of Bianchi and of Brans-Edgar-Ludwig, belong to a complete integrable system.

We emphasize that it is not so far known the deep and ultimate reason of this theorem, which represents an interesting research program in General Relativity [12]. Indeed, a profound analysis of this problem will probably lead to a better understanding of the gravitational field.

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