Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 97 (2018) 51-68 EISSN 2392-2192

Noether and matrix methods to construct local symmetries of Lagrangians

P. Lam-Estrada[1], G. Ovando[2], J. López-Bonilla[3,*], R. López-Vázquez[3], S. Vidal-Beltrán[3]

[1] Depto. de Matemáticas, ESFM-Instituto Politécnico Nacional, Edif. 9, Col. Lindavista CP 07738, CDMX, México

[2] CBI-Área de Física-AMA, UAM-Azcapotzalco, Av. San Pablo 180, Col. Reynosa-Tamps., CDMX, México

[3] ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, Col. Lindavista, CP 07738, CDMX, México

*E-mail address: jlopezb@ipn.mx

ABSTRACT

We exhibit an elementary presentation of matrix and Noether techniques to construct infinitesimal point symmetries of Lagrangians. Besides, we employ the Lanczos approach to Noether's theorem to obtain the first integral associated with each symmetry. We show applications to several singular Lagrangians of interest in classical mechanics.

***Keywords*:** Noether theorem, Singular Lagrangians, Local symmetries, Gauge identities

1. INTRODUCTION

We consider a physical system where the parameters $q_1, q_2, \ldots, q_n$ are its generalized coordinates, that is, there are n degrees of freedom. The action:

(Received 20 March 2018; Accepted 07 April 2018; Date of Publication 08 April 2018)

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)\, dt, \qquad \dot{q} = \frac{dq}{dt}, \tag{1}$$

is fundamental in the dynamical evolution of the system. We can change to new coordinates via the local transformations:

$$\tilde{t} = t + \varepsilon\, \alpha_0(t), \qquad \tilde{q}_i = q_i + \varepsilon\, \alpha_i(q,t), \qquad i = 1, 2, \dots, n \tag{2}$$

where: ε is an infinitesimal parameter, thus the action takes the value:

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t})\, d\tilde{t}. \tag{3}$$

If $\delta S = \delta\tilde{S}$, to first order in ε, then we say that the action is invariant under the transformations (2), that is, (2) are local symmetries of the Euler-Lagrange equations of motion.

The principal aim of this work is to show several techniques to investigate the existence of point symmetries for a given action, and to realize the explicit construction of the functions $\alpha_r,\ r = 0, \dots, n$. The Sec. 2 contains the method of Emmy Noether [1- 5] to achieve this aim. The Sec. 3 considers the particular (but important) case of $\alpha_0 = 0$ with $L(q,\dot{q})$, that is, when the time remains intact and L does not have explicit dependence in t. This particular situation is studied employing a matrix procedure [6-8] to obtain the so-called gauge identities [9], from which the coordinate transformations (local symmetries) leaving the action invariant, can be extracted. In Sec. 4 the Noether's theorem [10-13], in the Lanczos approach [14-16], is used to deduce the first integral associated with each point symmetry, thus we find that the 'conserved quantities' have relationship with the corresponding genuine constraints. We make applications to Lagrangians studied by several authors [4, 7, 17-20].

2. NOETHER'S METHOD

From (2) it is easy to deduce, to first order in ε, that:

$$L\left(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t}\right) d\tilde{t} = L\left(q, \frac{dq}{dt}, t\right) dt + \varepsilon\, N(q,\dot{q},t) dt, \tag{4}$$

with the Noether's function [4, 20]:

$$N = \frac{\partial}{\partial t}(L\alpha_0) + \frac{\partial L}{\partial q_i}\alpha_i + \frac{\partial L}{\partial \dot{q}_i}(\dot{\alpha}_i - \dot{q}_i\dot{\alpha}_0), \qquad \dot{\alpha}_i = \frac{\partial \alpha_i}{\partial t} + \frac{\partial \alpha_i}{\partial q_r}\dot{q}_r, \tag{5}$$

where the Dedekind (1868)-Einstein [21, 22] summation convention is used for repeated indices. The condition $\delta\tilde{S} = \delta S$ can be obtained if the variation of the Lagrangian is a total derivative, that is, if in (4):

$$N(q,\dot{q},t) = \frac{d}{dt}F(q,t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_a}\dot{q}_a, \tag{6}$$

thus in the left side of (6) the term without velocities is equal to $\frac{\partial F}{\partial t}$, the coefficient of $\dot{q}_a$ coincides with $\frac{\partial F}{\partial q_a}$, and each term nonlinear in the $\dot{q}_j$ must be zero, which leads to a set of partial differential equations (PDE) [4] for the functions $\alpha_0(t)$ and $\alpha_r(q,t)$.

We can apply the Noether's expressions to Lagrangians employed by several authors:

a). Rothe [7].

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \tag{7}$$

then (5, 6) imply the following set of PDE:

$$\frac{\partial \alpha_1}{\partial q_2} = 0, \qquad \frac{1}{2}(q_1 - q_2)^2 \dot{\alpha}_0 + q_2 \frac{\partial \alpha_1}{\partial t} + (q_1 - q_2)(\alpha_1 - \alpha_2) = \frac{\partial F}{\partial t},$$

$$\frac{\partial \alpha_1}{\partial q_1} - \frac{1}{2}\dot{\alpha}_0 = 0, \qquad q_2 \frac{\partial \alpha_1}{\partial q_2} = \frac{\partial F}{\partial q_2}, \qquad \alpha_2 + \frac{\partial \alpha_1}{\partial t} + q_2 \frac{\partial \alpha_1}{\partial q_1} = \frac{\partial F}{\partial q_1},$$

whose general solution is:

$$\alpha_0 = c_0, \qquad \alpha_2 = \alpha_1 - \dot{\alpha}_1, \qquad F = q_1 \alpha_1, \tag{8}$$

where: c_0 is any constant, $\alpha_1(t)$ is an arbitrary function, and we could ask the conditions $\alpha_1(t_j) = 0$, $j = 1, 2$. Thus (2) gives us the local symmetry:

$$\tilde{t} = t + \varepsilon\, c_0, \qquad \tilde{q}_1 = q_1 + \varepsilon\, \alpha_1, \qquad \tilde{q}_2 = q_2 + \varepsilon\, (\alpha_1 - \dot{\alpha}_1), \tag{9}$$

in according with the relations (2.5) in [7].

b). Henneaux [7, 17].

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2, \tag{10}$$

and from this Noether's procedure we deduce the system of PDE:

$$\frac{\partial \alpha_2}{\partial q_1} = \frac{\partial \alpha_3}{\partial q_1} = 0, \qquad \frac{\partial \alpha_2}{\partial q_3} + \frac{\partial \alpha_3}{\partial q_2} = 0, \qquad -\frac{\dot{\alpha}_0}{2} + \frac{\partial \alpha_2}{\partial q_2} = 0, \qquad -\frac{\dot{\alpha}_0}{2} + \frac{\partial \alpha_3}{\partial q_3} = 0,$$

$$\frac{\partial F}{\partial t} = \frac{1}{2}(q_2^2 + e^{2q_1})\dot{\alpha}_0 - e^{q_1}\frac{\partial \alpha_2}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial t} + e^{2q_1}\alpha_1 + q_2 \alpha_2, \qquad \frac{\partial F}{\partial q_1} = -e^{q_1}\frac{\partial \alpha_2}{\partial q_1} - q_2 \frac{\partial \alpha_3}{\partial q_1},$$

$$\frac{\partial F}{\partial q_2} = \frac{\partial \alpha_2}{\partial t} - e^{q_1}\frac{\partial \alpha_2}{\partial q_2} - q_2 \frac{\partial \alpha_3}{\partial q_2} - e^{q_1}\alpha_1, \qquad \frac{\partial F}{\partial q_3} = -e^{q_1}\frac{\partial \alpha_2}{\partial q_3} + \frac{\partial \alpha_3}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial q_3} - \alpha_2,$$

with the general solution:

$$\alpha_0 = c_0, \qquad \alpha_1 = e^{-q_1}\ddot{\alpha}_3, \qquad \alpha_2 = \dot{\alpha}_3, \qquad F = 0, \tag{11}$$

and the corresponding point symmetry have the structure:

$$\tilde{t} = t + \varepsilon\, c_0\,, \quad \tilde{q}_1 = q_1 + \varepsilon\, e^{-q_1}\ddot{\alpha}_3\,, \quad \tilde{q}_2 = q_2 + \varepsilon\, \dot{\alpha}_3\,, \quad \tilde{q}_3 = q_3 + \varepsilon\, \alpha_3\,, \quad (12)$$

where: $\alpha_3(t)$ is arbitrary, which are the expressions (2.7) in [7].

c). Torres del Castillo [19].

$$L = \frac{1}{6}\dot{q}^3 + \frac{1}{2}gt\dot{q}^2 - g^2 qt, \qquad g \text{ is a constant,} \quad (13)$$

therefore (5, 6) imply the PDE:

$$-\frac{1}{3}\dot{\alpha}_0 + \frac{1}{2}\frac{\partial\alpha}{\partial q} = 0, \quad gt\,\frac{\partial\alpha}{\partial t} = \frac{\partial F}{\partial q}, \quad -gt\dot{\alpha}_0 + \frac{\partial\alpha}{\partial t} + 2gt\frac{\partial\alpha}{\partial q} + g\alpha_0 = 0, \quad g^2(qt\dot{\alpha}_0 + q\alpha_0 + t\alpha) = -\frac{\partial F}{\partial t}$$

with the solution:

$$\alpha_0 = \frac{3}{2}\,t, \qquad \alpha = q - gt^2, \qquad F = g^2t^2\left(\frac{1}{4}\,gt^2 - 2q\right), \quad (14)$$

in harmony with the transformation (2) in [19] for the infinitesimal case. Thus the local symmetry is given by:

$$\tilde{t} = t + \frac{3}{2}\,\varepsilon\, t, \qquad \tilde{q} = q + \varepsilon\,(q - gt^2)\,. \quad (15)$$

d). Torres del Castillo [19].

$$L = \frac{1}{2}\,t^2\left(\dot{q}^2 - \frac{1}{3}\,q^6\right), \quad (16)$$

and this Noether's approach permits to obtain the set of PDE:

$$t^2\frac{\partial\alpha}{\partial t} = \frac{\partial F}{\partial q}, \qquad -\frac{1}{2}t^2\dot{\alpha}_0 + t^2\frac{\partial\alpha}{\partial q} + t\alpha_0 = 0, \qquad t\,q^5\left(\frac{1}{6}\,tq\dot{\alpha}_0 + \frac{1}{3}\,q\alpha_0 + t\alpha\right) = -\frac{\partial F}{\partial t}\,,$$

with the following solution:

$$\alpha_0 = 2t, \qquad \alpha = -q, \qquad F = 0, \quad (17)$$

equivalent to infinitesimal version of the transformation (4) in [19], and the point symmetry is:

$$\tilde{t} = t + 2\,\varepsilon\, t, \qquad \tilde{q} = q - \varepsilon\, q. \quad (18)$$

e). Havelková [4] – Torres del Castillo [20].

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3 \,, \tag{19}$$

its associated Noether's PDE system is:

$$\frac{\partial \alpha_1}{\partial q_2} = \frac{\partial \alpha_1}{\partial q_3} = \frac{\partial \alpha_3}{\partial q_1} = \frac{\partial \alpha_3}{\partial q_2} = 0, \qquad \frac{\partial \alpha_1}{\partial q_1} + \frac{\partial \alpha_3}{\partial q_3} - \dot{\alpha}_0 = 0, \qquad \frac{\partial F}{\partial q_3} = \frac{\partial \alpha_1}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial q_3} - \alpha_2 \,,$$

$$\frac{\partial F}{\partial t} = q_3(q_1 \dot{\alpha}_0 + \alpha_1) + q_1 \alpha_3 - q_2 \frac{\partial \alpha_3}{\partial t} \,, \qquad \frac{\partial F}{\partial q_1} = \frac{\partial \alpha_3}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial q_1} \,, \qquad \frac{\partial F}{\partial q_2} = -q_2 \frac{\partial \alpha_3}{\partial q_2} \,,$$

and the corresponding general solution is given in [4, 20]:

$$\alpha_0 = c_0, \quad \alpha_1 = -c_1 q_1 + c_2 e^t + c_3 e^{-t}, \quad \alpha_2 = -c_1 q_2 + f(q_3), \quad \alpha_3 = c_1 \, q_3 \,,$$
$$F = (c_2 \, e^t - c_3 \, e^{-t}) \, q_3 - \int^{q_3} f(u) du, \tag{20}$$

where: $f(q_3)$ is an arbitrary function and the c_j are constants. Thus, we have the local symmetry ([4] p. 28, and relations (30) in [20]):

$$\tilde{t} = t + \varepsilon \, c_0 \,, \qquad \tilde{q}_1 = q_1 + \varepsilon \, (-c_1 \, q_1 + c_2 \, e^t + c_3 \, e^{-t}),$$
$$\tilde{q}_2 = q_2 + \varepsilon \left(-c_1 \, q_2 + f(q_3)\right), \qquad \tilde{q}_3 = q_3 + \varepsilon \, c_1 \, q_3 \,. \tag{21}$$

f). Rothe [7].

$$L = \tfrac{1}{2}\dot{q}_1^2 + (q_2 - q_3) \, \dot{q}_1 + \tfrac{1}{2} \, (q_1 - q_2 + q_3)^2, \tag{22}$$

with the set of PDE:

$$\frac{\partial \alpha_1}{\partial q_2} = \frac{\partial \alpha_1}{\partial q_3} = 0, \qquad \frac{\partial \alpha_1}{\partial q_1} = \frac{1}{2} \, \dot{\alpha}_0 \,, \qquad \frac{\partial F}{\partial q_r} = (q_2 - q_3) \, \frac{\partial \alpha_1}{\partial q_r} \,, \quad r = 2, 3$$

$$\frac{\partial F}{\partial t} = (q_1 - q_2 + q_3) \left[\frac{1}{2} \, \dot{\alpha}_0 (q_1 - q_2 + q_3) + \alpha_1 - \alpha_2 + \alpha_3 \right] + (q_2 - q_3) \, \frac{\partial \alpha_1}{\partial t} \,,$$

$$\frac{\partial F}{\partial q_1} = \left(\frac{\partial \alpha_1}{\partial q_1} + \dot{\alpha}_0 \right) (q_2 - q_3) + \frac{\partial \alpha_1}{\partial t} + \alpha_2 - \alpha_3 \,,$$

which implies that $F = q_1 \, \alpha_1$ with the point symmetry:

$$\tilde{t} = t + \varepsilon \, c_0 \,, \quad \tilde{q}_1 = q_1 + \varepsilon \, \alpha_1 \,, \tilde{q}_2 = q_2 + \varepsilon \, (\alpha_1 - \dot{\alpha}_1 + \alpha_3), \quad \tilde{q}_3 = q_3 + \varepsilon \, \alpha_3 \,, \tag{23}$$

where: $\alpha_1(t)$ and $\alpha_3(t)$ are arbitrary functions, in according with the expressions (2.36) in [7].

In this Section the Noether's technique was applied to several Lagrangians to exhibit that the explicit construction of local symmetries it is equivalent to solve a set of PDE. The transformations (9, 12, 20, 23) have arbitrary functions, then the Sec. 3 shows the existence of one gauge identity by each arbitrary function present into the point symmetry, and also the presence of genuine constraints: We shall employ a matrix method [6-8, 23] for the particular case $\alpha_0 = 0$ with $L(q,\dot{q})$. Local symmetries of the action are not always easily detected; it is however crucial to unravel them since their knowledge is required for the quantization [7] of such singular systems. It is instructive to establish, in Sec. 3, a systematic method for detecting local symmetries on Lagrangian level. Such symmetries are intimately connected to the existence of so-called first class constraints, but the connection between point symmetries and first class constraints as generators of these symmetries is only manifest on Hamiltonian level [6, 7, 18, 23-25].

3. MATRIX ALGORITHM

Here we study the special transformations:

$$\tilde{t} = t, \qquad \tilde{q}_i = q_i + \varepsilon\, \alpha_i(q,t), \tag{24}$$

when: *t* is an ignorable variable, that is, the Lagrangian only depends of q_j and $\dot{q}_j$; then the variation of the action (1) is:

$$\delta S = -\int_{t_1}^{t_2} dt\ E_i^{(0)}\,\delta q_i = -\int_{t_1}^{t_2} dt\, \vec{E}^{(0)} \cdot \delta\vec{q}, \quad \delta q_i\big(t_j\big) = 0,\ \ j = 1,2 \tag{25}$$

with the zeroth level Euler derivatives:

$$E_i^{(0)}(q,\dot{q},\ddot{q},t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}, \qquad i = 1,2,\ldots,n \tag{26}$$

thus $\delta S = 0$ if :

$$\delta\vec{q} \cdot \vec{E}^{(0)} = -\frac{d(\varepsilon Q)}{dt}, \qquad Q(t_2) = Q(t_1) = 0, \tag{27}$$

without the participation of the equations of motion $E_i^{(0)} = 0$.

It is suitable to realize a splitting into (26) to show explicitly the presence of the accelerations:

$$\vec{E}^{(0)} = \left(E_i^{(0)}\right) = W^{(0)} \cdot \ddot{\vec{q}} + \vec{K}^{(0)}, \qquad \ddot{\vec{q}} = (\ddot{q}_j)_{n\,x\,1}, \tag{28}$$

with the Hessian matrix:

$$W^{(0)}(q,\dot{q}) = \left({W_{ij}}^{(0)}\right) = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)_{n\,x\,n} \tag{29}$$

and the zeroth level vector:

$$\vec{K}^{(0)}(q,\dot{q}) = \left({K_i}^{(0)}\right) = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\,\dot{q}_j - \frac{\partial L}{\partial q_i}\right)_{n\,x\,1}. \tag{30}$$

If p_0 is the rank [26-28] of $W^{(0)}$, then there exist $N_0 = n - p_0$ constraints, which on Lagrangian level present themselves as relations between coordinates q_i and velocities $\dot{q}_i$. We note that $N_0 =$ Nullity $W^{(0)}$, therefore there exist N_0 independent left zero mode eigenvectors $\vec{\omega}^{(0,k)}$ of $W^{(0)}$:

$$\vec{\omega}^{(0,k)} \cdot W^{(0)} = 0, \qquad r = 1, \dots, N_0 \tag{31}$$

and we can construct the quantities:

$$\Phi^{(0,r)} = \vec{\omega}^{(0,r)}(q,\dot{q}) \cdot \vec{E}^{(0)}(q,\dot{q},\ddot{q}) = \vec{\omega}^{(0,r)} \cdot \vec{K}^{(0)}, \tag{32}$$

which only depend on the coordinates and velocities, and vanish on the subspace of physical trajectories:

$$\Phi^{(0,r)}(q,\dot{q}) = 0, \qquad r = 1, \dots, N_0 \qquad \text{if} \qquad \vec{E}^{(0)} = \vec{0}, \tag{33}$$

we refer to them as the zero generation constraints.

Not all of the functions (32) may, however, be linearly independent. In this case one can find d_0 linear combinations of the zero mode eigenvectors:

$$\vec{v}^{(0,m_0)} = b_r{}^{(m_0)}\, \vec{\omega}^{(0,r)}, \qquad r = 1, \dots, N_0, \qquad m_0 = 1, \dots, d_0 \tag{34}$$

such that we obtain identically:

$$G^{(0,m_0)}(q,\dot{q}) = \vec{v}^{(0,m_0)} \cdot \vec{E}^{(0)} = \vec{v}^{(0,m_0)} \cdot \vec{K}^{(0)} = 0, \quad \textit{Gauge identities.} \tag{35}$$

In (34) we have d_0 linearly independent vectors into the subspace Kernel $W^{(0)}$ of dimension N_0 , then we construct $\bar{N}_0 = N_0 - d_0$ vectors $\vec{u}^{(0,\bar{m}_0)}(q,\dot{q})$ to establish a base for this subspace, and thus to introduce the quantities:

$$\varphi^{(0,\bar{m}_0)} = \vec{u}^{(0,\bar{m}_0)} \cdot \vec{E}^{(0)} = \vec{u}^{(0,\bar{m}_0)} \cdot \vec{K}^{(0)}, \qquad \bar{m}_0 = 1, \dots, \bar{N}_0 \,, \tag{36}$$

which vanish if $\vec{E}^{(0)} = \vec{0}$:

$$\varphi^{(0,\bar{m}_0)}(q,\dot{q}) = 0, \qquad \textit{Genuine constraints.} \tag{37}$$

The identities (35) imply that any variation of the form:

$$\delta\vec{q} = \varepsilon\, \beta_{m_0}(t)\vec{v}^{(0,m_0)}, \qquad m_0 = 1, \dots, d_0 \tag{38}$$

verifies (27) with $Q = 0$ and it will leave the action invariant.

Next we look for possible additional constraints by searching for further functions of the coordinates and velocities which vanish on the subspace of physical paths. To this effect we employ the following vector constructed from $\vec{E}^{(0)}$ and the time derivative of the non-trivial quantities (36):

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\varphi^{(0,1)} \\ \vdots \\ \frac{d}{dt}\varphi^{(0,\bar{N}_0)} \end{pmatrix}_{(n+\bar{N}_0)\,x\,1} = W^{(1)} \cdot \ddot{\vec{q}} + \vec{K}^{(1)}(q,\dot{q}), \tag{39}$$

a splitting similar to (28), where $W^{(1)}(q,\dot{q})$ is now the 'level 1' matrix:

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ \frac{\partial}{\partial \dot{q}_1}\varphi^{(0,1)} & \ldots & \frac{\partial}{\partial \dot{q}_n}\varphi^{(0,1)} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial \dot{q}_1}\varphi^{(0,\bar{N}_0)} & \ldots & \frac{\partial}{\partial \dot{q}_n}\varphi^{(0,\bar{N}_0)} \end{pmatrix}_{(n+\bar{N}_0)\,x\,n}, \quad \vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_j \frac{\partial}{\partial q_j}\varphi^{(0,1)} \\ \vdots \\ \dot{q}_j \frac{\partial}{\partial q_j}\varphi^{(0,\bar{N}_0)} \end{pmatrix}. \tag{40}$$

The constraints (37) hold for all times, then $\vec{E}^{(1)} = \vec{0}$ if $\vec{E}^{(0)} = \vec{0}$.

We next investigate for left zero modes of $W^{(1)}$, that is, eigenvectors $\vec{\omega}^{(1,k)}$ of its transpose with null eigenvalue, but now Nullity $W^{(1)\,T} = N_1 = n + \bar{N}_0 - p_1$ with $p_1 =$ rank $W^{(1)}$, $1 \le p_1 \le n$:

$$\vec{\omega}^{(1,r)} \cdot W^{(1)} = 0, \qquad r = 1, \ldots, N_1 \tag{41}$$

The proper vectors $\vec{\omega}^{(1,k)}$ include those of the previous level, augmented by an appropriate number of zeroes [28], which are denoted by $\vec{\omega}^{(1,N_1-N_0+1)}$, $\vec{\omega}^{(1,N_1-N_0+2)}, \ldots, \vec{\omega}^{(1,N_1)}$, they just reproduce the previous constraints and are therefore not considered. The remaining $N_1 - N_0 = \bar{N}_0 + p_0 - p_1 = n - p_1 - d_0$ zero modes (if they exist), when contracted with $\vec{E}^{(1)}$, lead to expressions at 'level 1':

$$\Phi^{(1,k)} = \vec{\omega}^{(1,k)} \cdot \vec{E}^{(1)} = \vec{\omega}^{(1,k)} \cdot \vec{K}^{(1)}, \qquad k = 1, \ldots, N_1 - N_0 \tag{42}$$

with the first generation constraints:

$$\Phi^{(1,k)}(q,\dot{q}) = 0, \tag{43}$$

on the subspace of physical trajectories ($\vec{E}^{(0)} = \vec{0}$). However, not all the quantities (42) may be linearly independent [the functions (36) also must be considered], then one can obtain d_1 new gauge identities at level 1 of the form:

$$G^{(1,m_1)} = \vec{v}^{(1,m_1)} \cdot \vec{E}^{(1)} - M^{m_1}_{\bar{m}_0}\, \varphi^{(0,\bar{m}_0)} \equiv 0, \quad \bar{m}_0 = 1, \ldots, \bar{N}_0, \quad m_1 = 1, \ldots, d_1 \tag{44}$$

with:

$$\vec{v}^{(1,m_1)} = c_r{}^{(m_1)}\, \vec{\omega}^{(1,r)}, \qquad r = 1, \ldots, N_1 - N_0 \tag{45}$$

In the subspace generated by $\vec{\omega}^{(1,k)},\ k = 1, \dots, N_1 - N_0$ we have the vectors (45), then into it we construct $\bar{N}_1 = N_1 - N_0 - d_1$ vectors $\vec{u}^{(1,\bar{m}_1)}(q, \dot{q})$ to complete a base, and we introduce the quantities:

$$\varphi^{(1,\bar{m}_1)} = \vec{u}^{(1,\bar{m}_1)} \cdot \vec{E}^{(1)} = \vec{u}^{(1,\bar{m}_1)} \cdot \vec{K}^{(1)}, \qquad \bar{m}_1 = 1, \dots, \bar{N}_1 \tag{46}$$

which represent, if $\vec{E}^{(0)} = \vec{0}$, genuine new constraints at level 1:

$$\varphi^{(1,\bar{m}_1)}(q, \dot{q}) = 0. \tag{47}$$

We now adjoin the new gauge identities (44) to the previous identities (35). With the functions (46) we proceed as before, adjoining their time derivative to (39), and construct $W^{(2)}$ as well as $\vec{K}^{(2)}$. This iterative process will terminate when there are no further zero modes [dim (Kernel $W^{(j)}) = 0$ for some value of j] or if the constraints generated are linear combinations of the previous ones, and hence lead to gauge identities only. At this point the algorithm has unraveled all the constraints of the Euler-Lagrange equations of motion.

We shall apply this matrix method to Lagrangians studied in Sec. 2, to determine the corresponding local symmetry transformations for the case (24) when t is an ignorable variable. Thus we will see that each element of the maximal set of linearly independent gauge identities can be multiplied by an arbitrary function and the sum of these expressions permits to obtain the point symmetries, hence the number of independent arbitrary functions equals the number of gauge identities generated by the algorithm. This procedure can be carried out completely without the need of developing the Dirac-Hamilton formalism [7, 18, 24].

i). Lagrangian (7):

$$L = \tfrac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \tfrac{1}{2}(q_1 - q_2)^2, \qquad n = 2,$$

$\vec{E}^{(0)}$ adopts the form (28) with:

$$W^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_0 = 1, \quad \vec{K}^{(0)} = \begin{pmatrix} \dot{q}_2 + q_2 - q_1 \\ -q_2 - \dot{q}_1 + q_1 \end{pmatrix}, \quad N_0 = n - p_0 = 1, \tag{48}$$

and we have one constraint on Lagrangian level 0, then from (31, 32):

$$\vec{\omega}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi^{(0,1)} = {E_2}^{(0)}, \quad d_0 = 0 \quad \therefore \quad \vec{u}^{(0,1)} = \vec{\omega}^{(0,1)}, \tag{49}$$

without gauge identities:

$$\varphi^{(0,1)} = \Phi^{(0,1)} = {E_2}^{(0)} = -q_2 - \dot{q}_1 + q_1, \qquad \bar{N}_0 = 1, \tag{50}$$

which gives us the genuine constraint $\varphi^{(0,1)} = 0$ on the subspace of physical paths. At the level 1, from (39, 40):

$$W^{(1)} = \begin{pmatrix} W^{(0)} \\ -1 \quad 0 \end{pmatrix}, \qquad p_1 = 1, \qquad \vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix}, \qquad N_1 = 2, \tag{51}$$

and (41) implies two zero modes:

$$\vec{\omega}^{(1,1)} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \vec{\omega}^{(1,2)} = \begin{pmatrix}\vec{\omega}^{(0,1)}\\0\end{pmatrix}, \quad \Phi^{(1,2)} = \begin{pmatrix}\vec{\omega}^{(0,1)}\\0\end{pmatrix}\cdot\begin{pmatrix}\vec{E}^{(0)}\\ \frac{d}{dt}\varphi^{(0,1)}\end{pmatrix} = \varphi^{(0,1)}, \quad (52)$$

then the remaining quantity is:

$$\Phi^{(1,1)} = \vec{\omega}^{(1,1)}\cdot\vec{E}^{(1)} = {E_1}^{(1)} + {E_3}^{(1)} = {E_1}^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = {E_1}^{(0)} + \frac{d}{dt}{E_2}^{(0)}$$
$$= q_2 + \dot{q}_1 - q_1, \quad (53)$$

but it is not independent because $\Phi^{(1,1)} = -\varphi^{(0,1)}$, that is:

$${E_1}^{(0)} + {E_2}^{(0)} + \tfrac{d}{dt}{E_2}^{(0)} = 0, \qquad d_1 = 1, \qquad \bar{N}_1 = 0, \quad (54)$$

thus the matrix process terminates here at level 1 because it leads to gauge identities only.

We multiply (54) by $\varepsilon\,\alpha_1(t)$ to obtain:

$$\varepsilon\,\alpha_1\,{E_1}^{(0)} + \varepsilon\,(\alpha_1 - \dot{\alpha}_1){E_2}^{(0)} = -\frac{d}{dt}\left(\varepsilon\,\alpha_1 {E_2}^{(0)}\right),$$

then the comparison with (27) gives $\delta q_1 = \varepsilon\,\alpha_1$ and $\delta q_2 = \varepsilon\,(\alpha_1 - \dot{\alpha}_1)$, in harmony with the local symmetry transformations (8) for the case $\alpha_0 = 0$.

ii). Lagrangian (10):

$$L = \tfrac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \tfrac{1}{2}(\dot{q}_3 - q_2)^2\,, \qquad n = 3,$$

$\vec{E}^{(0)}$ has the structure (28) such that:

$$W^{(0)} = \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad p_0 = 2, \vec{K}^{(0)} = \begin{pmatrix}e^{q_1}(\dot{q}_2 - e^{q_1})\\ -e^{q_1}\dot{q}_1 + \dot{q}_3 - q_2\\ -\dot{q}_2\end{pmatrix}, \quad N_0 = 1, \quad (55)$$

with one constraint at level 0, without gauge identities ($d_0 = 0$, $\bar{N}_0 = 1$):

$$\vec{\omega}^{(0,1)} = \vec{u}^{(0,1)} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \Phi^{(0,1)} = \varphi^{(0,1)} = {E_1}^{(0)} = e^{q_1}(\dot{q}_2 - e^{q_1}). \quad (56)$$

At the level 1:

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ 0 & e^{q_1} & 0\end{pmatrix}, \qquad p_1 = 2, \qquad \vec{K}^{(1)} = \begin{pmatrix}\vec{K}^{(0)}\\ \dot{q}_1 - \dot{q}_2\end{pmatrix}, \qquad N_1 = 2,$$

$$\vec{\omega}^{(1,1)} = \begin{pmatrix} 0 \\ -e^{q_1} \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \qquad \Phi^{(1,2)} = \varphi^{(0,1)}, \tag{57}$$

and

$$\Phi^{(1,1)} = -e^{q_1}{E_2}^{(1)} + {E_4}^{(1)} = -e^{q_1}{E_2}^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = -e^{q_1}{E_2}^{(0)} + \frac{d}{dt}{E_1}^{(0)},$$
$$= e^{q_1}(-e^{q_1}\dot{q}_1 - \dot{q}_3 + q_2 + \dot{q}_1\dot{q}_2) = \dot{q}_1\varphi^{(0,1)} + e^{q_1}(q_2 - \dot{q}_3),$$

where the first term is proportional to (56), then we can continue the process with the second term:

$$\varphi^{(1,1)} = e^{q_1}(q_2 - \dot{q}_3) = \Phi^{(1,1)} - \dot{q}_1\varphi^{(0,1)} = -\dot{q}_1{E_1}^{(0)} - e^{q_1}{E_2}^{(0)} + \frac{d}{dt}{E_1}^{(0)}. \tag{58}$$

At the level 2:

$$W^{(2)} = \begin{pmatrix} & W^{(1)} & \\ 0 & 0 & -e^{q_1} \end{pmatrix}, \qquad p_2 = 2, \qquad \vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ e^{q_1}(\dot{q}_1 q_2 + \dot{q}_2 - \dot{q}_1\dot{q}_3) \end{pmatrix},$$

$$\vec{\omega}^{(2,1)} = \begin{pmatrix} 0 \\ 0 \\ e^{q_1} \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{\omega}^{(2,2)} = \begin{pmatrix} \vec{\omega}^{(1,1)} \\ 0 \end{pmatrix}, \qquad \vec{\omega}^{(2,3)} = \begin{pmatrix} \vec{\omega}^{(1,2)} \\ 0 \end{pmatrix}, \tag{59}$$

$$\Phi^{(2,1)} = \vec{\omega}^{(2,1)} \cdot \vec{E}^{(2)} = e^{q_1}{E_3}^{(2)} + {E_5}^{(2)} = e^{q_1}{E_3}^{(1)} + \frac{d}{dt}\varphi^{(1,1)} = \dot{q}_1 e^{q_1}(q_2 - \dot{q}_3)$$
$$= \dot{q}_1\varphi^{(1,1)},$$

which implies the gauge identity:

$$\dot{q}_1^2{E_1}^{(0)} + \dot{q}_1 e^{q_1}{E_2}^{(0)} + e^{q_1}{E_3}^{(0)} - \dot{q}_1\frac{d}{dt}{E_1}^{(0)} + \frac{d}{dt}\left(-\dot{q}_1{E_1}^{(0)} - e^{q_1}{E_2}^{(0)} + \frac{d}{dt}{E_1}^{(0)}\right)$$
$$= 0. \tag{60}$$

We multiply (60) by $\varepsilon\,\alpha_3(t)e^{-q_1}$ to deduce the expression:

$$\varepsilon\,\ddot{\alpha}_3 e^{-q_1}{E_1}^{(0)} + \varepsilon\,\dot{\alpha}_3{E_2}^{(0)} + \varepsilon\,\alpha_3{E_3}^{(0)} = \varepsilon\,\frac{d}{dt}[\alpha_3(\dot{q}_3 - q_2) + \dot{\alpha}_3(\dot{q}_2 - e^{q_1})],$$

whose comparison with (27) permits to obtain the point symmetry transformations (12) $\delta q_1 = \varepsilon\,\ddot{\alpha}_3 e^{-q_1}$, $\delta q_2 = \varepsilon\,\dot{\alpha}_3$ and $\delta q_3 = \varepsilon\,\alpha_3$ for $\alpha_0 = 0$.

iii). Lagrangian (19):

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3, \qquad n = 3.$$

In this case:

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad p_0 = 2, \qquad \vec{K}^{(0)} = \begin{pmatrix} -q_3 \\ \dot{q}_3 \\ -\dot{q}_2 - q_1 \end{pmatrix}, \qquad N_0 = 1,$$

$$\vec{\omega}^{(0,1)} = \vec{u}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Phi^{(0,1)} = \varphi^{(0,1)} = E_2{}^{(0)} = \dot{q}_3,$$

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ 0 & 0 & 1 \end{pmatrix}, \quad p_1 = 2, \tag{61}$$

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ 0 \end{pmatrix}, \qquad \vec{\omega}^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \qquad \Phi^{(1,2)} = \varphi^{(0,1)},$$

then:

$$\Phi^{(1,1)} = E_1{}^{(1)} - E_4{}^{(1)} = E_1{}^{(0)} - \tfrac{d}{dt} E_2{}^{(0)} = -q_3 = \varphi^{(1,1)}, \tag{62}$$

and it is necessary to go towards level 2:

$$W^{(2)} = \begin{pmatrix} & W^{(1)} & \\ 0 & 0 & 0 \end{pmatrix}, \; p_2 = 2, \; \vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ -\dot{q}_3 \end{pmatrix}, \; \vec{\omega}^{(2,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; \vec{\omega}^{(2,2)} = \begin{pmatrix} \vec{\omega}^{(1,1)} \\ 0 \end{pmatrix},$$
$$\vec{\omega}^{(2,3)} = \begin{pmatrix} \vec{\omega}^{(1,2)} \\ 0 \end{pmatrix}, \tag{63}$$

such that:

$$\Phi^{(2,1)} = E_5{}^{(2)} = \frac{d}{dt}\varphi^{(1,1)} = \frac{d}{dt}(E_1{}^{(0)} - \frac{d}{dt}E_2{}^{(0)}) = -\dot{q}_3 = -\varphi^{(0,1)} = -E_2{}^{(0)},$$

thus we have the gauge identity:

$$E_2{}^{(0)} + \tfrac{d}{dt}\left(E_1{}^{(0)} - \tfrac{d}{dt}E_2{}^{(0)}\right) = 0. \tag{64}$$

We multiply (64) by $\varepsilon\,\alpha$ to obtain the relation:

$$\varepsilon\,\dot{\alpha} E_1{}^{(0)} + \varepsilon\,(\ddot{\alpha} - \alpha)\,E_2{}^{(0)} = \varepsilon\,\frac{d}{dt}\left[\dot{\alpha}\,E_2{}^{(0)} + \alpha\left(E_1{}^{(0)} - \frac{d}{dt}E_2{}^{(0)}\right)\right],$$

then from (27):

$$\delta q_1 = \varepsilon\,\dot{\alpha}, \qquad \delta q_2 = \varepsilon\,(\ddot{\alpha} - \alpha), \qquad \delta q_3 = 0, \tag{65}$$

and, for example, if $\alpha = c_2\, e^{t} - c_3\, e^{-t}$ we deduce (20) in the particular case $c_1 = f = 0$.

iv). Lagrangian (22):

$$L = \tfrac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\,\dot{q}_1 + \tfrac{1}{2}\,(q_1 - q_2 + q_3)^2, \qquad n = 3.$$

At the level 0:

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p_0 = 1, \quad \vec{K}^{(0)} = \begin{pmatrix} \dot{q}_2 - \dot{q}_3 - q_1 + q_2 - q_3 \\ -\dot{q}_1 + q_1 - q_2 + q_3 \\ \dot{q}_1 - q_1 + q_2 - q_3 \end{pmatrix},$$

$$\vec{\omega}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{\omega}^{(0,2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\Phi^{(0,1)} = {E_2}^{(0)} = -\dot{q}_1 + q_1 - q_2 + q_3, \quad \Phi^{(0,2)} = {E_3}^{(0)} = \dot{q}_1 - q_1 + q_2 - q_3 = -\Phi^{(0,1)},$$

with the gauge identity $\Phi^{(0,1)} + \Phi^{(0,2)} = 0$, that is:

$$G^{(0,1)} = \left(\vec{\omega}^{(0,1)} + \vec{\omega}^{(0,2)}\right)\cdot \vec{E}^{(0)} = \vec{v}^{(0,1)} \cdot \vec{E}^{(0)} = {E_2}^{(0)} + {E_3}^{(0)} = 0, \quad \vec{v}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{66}$$

and we may employ the vector:

$$\vec{u}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \therefore \qquad \varphi^{(0,1)} = {E_2}^{(0)}.$$

At level 1:

$$W^{(1)} = \begin{pmatrix} W^{(0)} \\ -1 \quad 0 \quad 0 \end{pmatrix}, \qquad p_1 = 1,$$

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_1 - \dot{q}_2 + \dot{q}_3 \end{pmatrix}, \vec{\omega}^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \vec{\omega}^{(1,3)} = \begin{pmatrix} \vec{\omega}^{(0,2)} \\ 0 \end{pmatrix},$$

$$\Phi^{(1,1)} = {E_1}^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = {E_1}^{(0)} + \frac{d}{dt}{E_2}^{(0)} = \dot{q}_1 - q_1 + q_2 - q_3 = -\varphi^{(0,1)} = -{E_2}^{(0)},$$

with the gauge identity:

$${E_1}^{(0)} + {E_2}^{(0)} + \tfrac{d}{dt}{E_2}^{(0)} = 0. \tag{67}$$

We multiply (66) and (67) by $\varepsilon\,\alpha_3$ and $\varepsilon\,\alpha_1$, respectively, and we add the corresponding expressions to obtain:

$$\varepsilon\,\alpha_1\,{E_1}^{(0)} + \varepsilon\,(\alpha_3 - \dot{\alpha}_1 + \alpha_1)\,{E_2}^{(0)} + \varepsilon\,\alpha_3\,{E_3}^{(0)} = -\frac{d}{dt}\left(\varepsilon\,\alpha_1\,{E_2}^{(0)}\right),$$

and thus from (27) we reproduce the local symmetry transformations (23) because $\delta q_1 = \varepsilon\,\alpha_1$, $\delta q_2 = \varepsilon\,(\alpha_1 - \dot{\alpha}_1 + \alpha_3)$ and $\delta q_3 = \varepsilon\,\alpha_3$.

In this Section the matrix method was applied to several Lagrangians studied in Sec. 2 to show its compatibility with the procedure introduced by Noether. In the next Section we use the Lanczos technique [14-16] to deduce the conservation laws [10, 13, 29] associated with point symmetries, and we find that the corresponding conserved quantities have connection with the genuine constraints for $\vec{E}^{(0)} = \vec{0}$.

4. LANCZOS APPROACH TO NOETHER'S THEOREM

Noether [1-3, 10, 13] proved that in a variational principle the existence of symmetries implies the presence of conservation laws. Lanczos [14-16] uses this Noether's result in the following manner:

1) We consider a global symmetry with constant parameters.
2) After we accept that the parameters are functions of t, that is, now the transformation is a local symmetry
3) Into Lagrangian we substitute this local mapping to first order in ε, and the parameters are new degrees of freedom.
4) Then the Euler-Lagrange equations for these parameters give us the conservation laws.

Here we apply this Lanczos approach to Lagrangians from Secs. 2 and 3, and we find that the conserved quantities are in terms of the genuine constraints on the subspace of physical trajectories.

A). Lagrangian (7):

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \tag{68}$$

First, in the transformation (9) we employ $c_0 = 0$ with the constant $\alpha_1 = a$, thus we have the global symmetry $\tilde{t} = t$, $\tilde{q}_1 = q_1 + \varepsilon\,a$, $\tilde{q}_2 = q_2 + \varepsilon\,a$. Now we change the parameter a by the function $\beta(t)$, our new degree of freedom, to obtain the local symmetry $\tilde{t} = t$, $\tilde{q}_1 = q_1 + \varepsilon\,\beta(t)$, $\tilde{q}_2 = q_2 + \varepsilon\,\beta(t)$, therefore $\tilde{L} = L + \varepsilon\left[\beta\,\dot{q}_1 + \dot{\beta}\,(\dot{q}_1 + q_2)\right]$, and from the Euler-Lagrange equation for β, $\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\beta}}\right) - \frac{\partial \tilde{L}}{\partial \beta} = 0$, we deduce that:

$$\frac{d}{dt}(\dot{q}_1 + q_2 - q_1) = \frac{d}{dt}\varphi^{(0,1)} = 0, \tag{69}$$

which is correct for the genuine constraints because $\vec{E}^{(0)} = \vec{0}$. The expression (69) is the conserved quantity associated with (9) for $c_0 = 0$.

The transformation (9) for $\alpha_1 = 0$ is the global symmetry $\tilde{t} = t + \varepsilon\, c_0$, $\tilde{q}_1 = q_1$, $\tilde{q}_2 = q_2$, and now we consider that c_0 is the function $\beta(t)$, then $\tilde{L}\, d\tilde{t} = \left[L + \dot{\beta}(L - \dot{q}_1^2 - \dot{q}_1 q_2)\right] dt$ and the Euler-Lagrange equation for β implies the conservation of the Hamiltonian function $H = \dot{q}_1(\dot{q}_1 + q_2) - L$, because t is ignorable in (68).

B). Lagrangian (10):

$$L = \tfrac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \tfrac{1}{2}(\dot{q}_3 - q_2)^2\,. \tag{70}$$

In the transformation (12) we utilize $c_0 = 0$ and $\alpha_3 = a =$ constant, to obtain the global symmetry $\tilde{t} = t$, $\tilde{q}_1 = q_1$, $\tilde{q}_2 = q_2$, $\tilde{q}_3 = q_3 + \varepsilon\, a$. In (70) we apply this symmetry when $a \rightarrow \beta(t)$:

$$\tilde{L} = L + \varepsilon\, \dot{\beta}\, (\dot{q}_3 - q_2) \qquad \therefore \qquad \frac{d}{dt}(\dot{q}_3 - q_2) = \frac{d}{dt}\varphi^{(1,1)} = 0, \tag{71}$$

on the subspace of physical paths. If it is necessary, in (12) we can use $\alpha_3 = 0$ with $c_0 \neq 0$ to deduce the conservation of the corresponding Hamiltonian because (70) has not explicit dependence of t.

C). Lagrangian (19):

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3\,, \tag{72}$$

the transformations (20) permit several situations, in fact

C1). $c_0 = c_2 = c_3 = f = 0$, $c_1 \neq 0$, then $\tilde{L} = L + \varepsilon\, \dot{\beta}\, (q_3\dot{q}_1 - q_3 q_2 - \dot{q}_3 q_1)$ and the Lanczos procedure implies:

$$\text{Constant} = q_1\, \dot{q}_3 - (\dot{q}_1 - q_2) q_3 = q_1\, \varphi^{(0,1)} + (\dot{q}_1 - q_2)\, \varphi^{(1,1)}, \tag{73}$$

whose value is zero because $\vec{E}^{(0)} = \vec{0}$.

C2). $c_r = 0$, $r = 0, \ldots, 3$ and $f = a$, therefore $\tilde{t} = t$, $\tilde{q}_1 = q_1$, $\tilde{q}_2 = q_2 + \varepsilon\, a$, $\tilde{q}_3 = q_3$.

If $a \rightarrow \beta(t)$, from (72) we have that $\tilde{L} = L + \varepsilon \left[\dot{\beta}\, \dot{q}_3 + \beta(q_3 - \dot{q}_3)\right] e^{-t}$ and the Euler-Lagrange equation for β leads to

$$\ddot{q}_3 - q_3 = \frac{d}{dt}\varphi^{(0,1)} + \varphi^{(1,1)} = 0, \tag{74}$$

on the subspace of physical trajectories.

C3). $c_0 = c_1 = c_3 = f = 0$, $c_2 \neq 0$, then $\tilde{L} = L + \varepsilon \left[\dot{\beta}\, \dot{q}_3 + \beta\, (q_3 + \dot{q}_3)\right] e^{t}$ and the Lanczos approach gives (74) again.

D). Lagrangian (22):

$$L = \frac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\,\dot{q}_1 + \frac{1}{2}\,(q_1 - q_2 + q_3)^2. \tag{75}$$

In (23) the first option is $c_0 = \alpha_3 = 0 \;\therefore\; \tilde{t} = t,\ \tilde{q}_1 = q_1 + \varepsilon\, a,\ \tilde{q}_2 = q_2 + \varepsilon\, a,$ $\tilde{q}_3 = q_3$. If $a \to \beta(t)$, then from (75) $\tilde{L} = L + \varepsilon\left[\dot{\beta}\,(\dot{q}_1 + q_2 - q_3) + \beta\,\dot{q}_1\right]$, thus:

$$\frac{d}{dt}(\dot{q}_1 + q_2 - q_3 - q_1) = \frac{d}{dt}\varphi^{(0,1)} = 0, \qquad \vec{E}^{(0)} = \vec{0}. \tag{76}$$

The second case is $c_0 = \alpha_1 = 0,\ \tilde{t} = t,\ \tilde{q}_1 = q_1,\ \tilde{q}_2 = q_2 + \varepsilon\, a,\ \tilde{q}_3 = q_3 + \varepsilon\, a,$ and if $a \to \beta(t)$ we obtain $\tilde{L} = L \;\therefore$ the Euler-Lagrange equation for β gives $0 = 0$.

5. CONCLUSIONS

The relations (69, 71, 73, 74, 76) show that the conservation laws (deduced via Lanczos technique) associated with point symmetries have relationship with the genuine constraints on the subspace of physical paths. It is interesting to comment that Jacobi [30, 31] had already found a connection between the Euclidean invariance of the mechanical Lagrangian and the conservation laws for linear and angular momenta; Jacobi was the first person who derived the 10 integrals of the mechanical equations of motion, (partly) by using the infinitesimal transformations contained in the Euclidean group. Schütz [32] derived the energy conservation from the symmetry principle. Hamel [33] proceeded systematically to exploit the connection between symmetry transformations and conserved quantities in mechanics, he extensively used Lie's work on continuous groups. Herglotz [34] discussed the ten-parameter Poincaré group of 'motions' in the four-dimensional space, and related its symmetries to ten general integrals, among which the most familiar are the energy, linear momentum and angular momentum; the other three conserved quantities correspond to a generalization of the center-of-mass integrals in classical mechanics. The extension of Noether's theorem to quantum mechanics was made by Wigner [35].

References

[1] E. Noether, *Nachr. Ges. Wiss. Göttingen* 2 (1918) 235-257.

[2] D. E. Neuenschwander, *Emmy Noether's wonderful theorem*, The Johns Hopkins University Press, Baltimore, USA (2011).

[3] Y. Kosmann-Schwarzbach, *The Noether theorems*, Springer, New York (2011).

[4] Monika Havelková, *Comm. in Maths.* 20(1) (2012) 23-32.

[5] P. Lam-Estrada, J. López-Bonilla, R. López-Vázquez, G. Ovando, *Operations Research and Applications Journal (India)* 2(1) (2015) 1-8.

[6] E. C. G. Sudarshan, N. Mukunda, *Classical dynamics: A modern perspective*, John Wiley and Sons, New York (1974).

[7] H. J. Rothe, K. D. Rothe, *Classical and quantum dynamics of constrained Hamiltonian systems*, World Scientific Lecture Notes in Physics 81, Singapore (2010).

[8] P. Lam-Estrada, J. López-Bonilla, R. López-Vázquez, G. Ovando, *Bull. of Kerala Maths. Assoc.* 12(1) (2015) 43-52.

[9] P. Lam-Estrada, J. López-Bonilla, R. López-Vázquez, G. Ovando, *Prespacetime Journal* 6(3) (2015) 238-246.

[10] Nina Byers, *Emmy Noether's discovery of the deep connection between symmetries and conservation laws*, Proc. Symp. Heritage E. Noether, Bar Ilan University, Tel Aviv, Israel, Dec. 2-3 (1996).

[11] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*. *Its role in Mathematics and Physics*, D. Van Nostrand, London (1966).

[12] D. Lovelock, H. Rund, *Tensors, differential forms and variational principles*, Dover-New York (1989).

[13] P. Lam-Estrada, J. López-Bonilla, R. López-Vázquez, *Prespacetime Journal* 6(4) (2015) 322-325.

[14] C. Lanczos, *The variational principles of mechanics*, University of Toronto Press (1970).

[15] C. Lanczos, *Bull. Inst. Math. Appl.* 9(8) (1973) 253-258.

[16] I. Guerrero-Moreno, J. López-Bonilla, R. López-Vázquez, *Prespacetime Journal* 7(13) (2016) 1789-1792.

[17] M. Henneaux, C. Teitelboim, J. Zanelli, *Nucl. Phys. B* 332(1) (1990) 169-188.

[18] M. Henneaux, C. Teitelboim, *Quantization of gauges systems*, Princeton University Press, New Jersey (1994).

[19] G. F. Torres del Castillo, C. Andrade-Mirón, R. Bravo-Rojas, *Rev. Mex. Fís. E* 59 (2013) 140-147.

[20] G. F. Torres del Castillo, *Rev. Mex. Fís*. 60 (2014) 129-135.

[21] M. A. Sinaceur, *Rev. Hist. Sci*. 43 (1990) 221-294.

[22] D. Laugwitz, *Bernhard Riemann* 1826-1866. *Turning points in the conception of mathematics*, Birkhäuser, Boston, USA (2008).

[23] A. Shirzad, *J. Phys. A: Math. Gen*. 31(11) (1998) 2747-2760.

[24] P.A.M. Dirac, *Lectures on quantum mechanics*, Yeshiva University, New York (1964).

[25] M. Chaichian, D. Louis-Martinez, *J. Math. Phys.* 35(12) (1994) 6536-6545.

[26] L. J. Gerstein, *Am. Math. Monthly* 95(10) (1988) 950-952.

[27] R. A. Horn, Ch. R. Johnson, *Matrix analysis*, Cambridge University Press (1990).

[28] G. Bahadur Thapa, P. Lam-Estrada, J. López-Bonilla, *World Scientific News* 95 (2018) 100-110.

[29] J. López-Bonilla, J. Morales, G. Ovando, *J. Zhejiang Univ. Sci. A* 8(4) (2007) 665-668.

[30] C. G. J. Jacobi, *Vorlesungen über dynamik* (1842-43) [First edited in 1866 by Clebsch].

[31] J. Mehra, *Einstein, Hilbert, and the theory of gravitation*, D. Reidel Pub., Boston, USA (1974).

[32] J. R. Schütz, *Nachr. Ges. Wiss. Göttingen* (1897) 110.

[33] G. Hamel, *Math. Ann.* 59 (1904) 416-434.

[34] G. Herglotz, *Ann. der Physik* 36 (1911) 493-517.

[35] E. P. Wigner, *Nachr. Ges. Wiss. Göttingen* (1927) 375-381.