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SHORT COMMUNICATION

Vector field in the reciprocal space

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ABSTRACT

As a preparation for the study of Maxwell equations under a spatial Fourier transform, we analyze properties of the transverse and longitudinal components of an arbitrary vector field in the reciprocal space.

***Keywords*:** Transverse and longitudinal δ-function, Reciprocal Space, Spatial Fourier transformation

1. INTRODUCTION

For the vector field $\vec{E}(\vec{r})$, its Fourier transformation is given by [1, 2]:

$$\mathcal{F}\left[\vec{E}(\vec{r})\right] = \boldsymbol{\mathcal{E}}\left(\vec{k}\right) = \frac{1}{(2\pi)^{3/2}} \int d^3r \; e^{-i\,\vec{k}\cdot\vec{r}}\, \vec{E}(\vec{r}), \tag{1}$$

that in the reciprocal space accepts a splitting in longitudinal and transverse components [3]:

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$$\boldsymbol{\mathcal{E}}(\vec{k}) = \boldsymbol{\mathcal{E}}^{\parallel}(\vec{k}) + \boldsymbol{\mathcal{E}}^{\perp}(\vec{k}), \tag{2}$$

where: $\boldsymbol{\mathcal{E}}^{\parallel} \propto \vec{k}$ and $\mathcal{E}^{\perp} \perp \vec{k}$, that is:

$$\vec{k} \cdot \boldsymbol{\mathcal{E}}^{\perp} = 0, \qquad \vec{k} \times \boldsymbol{\mathcal{E}}^{\parallel} = \vec{0}. \tag{3}$$

People moving at different velocities would separate the same field into transverse and longitudinal components in a different way [4].

On the other hand, from (1) it is immediate the corresponding inverse Fourier transform:

$$\mathcal{F}^{-1}[\boldsymbol{\mathcal{E}}(\vec{k})] = \vec{E}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{i\,\vec{k}\cdot\vec{r}} \, \boldsymbol{\mathcal{E}}(\vec{k}), \tag{4}$$

therefore:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{i\,\vec{k}\cdot\vec{r}} \, i\,\vec{k} \cdot \boldsymbol{\mathcal{E}}, \qquad \vec{\nabla} \times \vec{E} = \frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{i\,\vec{k}\cdot\vec{r}} \, i\,\vec{k} \times \boldsymbol{\mathcal{E}}, \tag{5}$$

hence:

$$\mathcal{F}[\vec{\nabla} \cdot \vec{E}] = i\,\vec{k} \cdot \boldsymbol{\mathcal{E}}, \qquad \mathcal{F}[\vec{\nabla} \times \vec{E}] = i\,\vec{k} \times \boldsymbol{\mathcal{E}}. \tag{6}$$

The decomposition (2) means:

$$\vec{E} = \vec{E}^{\parallel} + \vec{E}^{\perp}, \tag{7}$$

such that $\vec{E}^{\parallel} = \mathcal{F}^{-1}[\boldsymbol{\mathcal{E}}^{\parallel}]$ and $\vec{E}^{\perp} = \mathcal{F}^{-1}[\boldsymbol{\mathcal{E}}^{\perp}]$; we note that $\boldsymbol{\mathcal{E}}^{\parallel} \cdot \boldsymbol{\mathcal{E}}^{\perp} = 0$, but in general $\vec{E}^{\parallel} \cdot \vec{E}^{\perp} \neq 0$. Then (3) is equivalent to:

$$\vec{\nabla} \cdot \vec{E}^{\perp} = 0, \qquad \vec{\nabla} \times \vec{E}^{\parallel} = \vec{0}, \tag{8}$$

which represents the known Helmholtz theorem [5, 6].

In this work we study properties of the transverse and longitudinal components of $\vec{E}$ and $\boldsymbol{\mathcal{E}}$, which are important in the analysis of Maxwell equations because $\vec{E}$ can be an electric or magnetic field or the electromagnetic potential verifying certain gauge [1, 7-10].

2. CONVOLUTION AND PARSEVAL-PLANCHEREL IDENTITY

We have the Parseval-Plancherel's relation:

$$\int d^3r \ f^*(\vec{r}) \, g(\vec{r}) = \int d^3k \ \mathcal{F}[f](\vec{k})^* \, \mathcal{F}[g](\vec{k}), \tag{9}$$

which implies the following identities [1] for arbitrary vector fields:

$$\int d^3r\ \vec{M}^* \cdot \vec{N} = \int d^3k\ \mathcal{F}[\vec{M}]^* \cdot \mathcal{F}[\vec{N}], \quad \int d^3r\ \vec{M}^* \times \vec{N} = \int d^3k\ \mathcal{F}[\vec{M}]^* \times \mathcal{F}[\vec{N}], \tag{10}$$

for example, if $\vec{M} = \vec{E}^{\parallel}$ and $\vec{N} = \vec{B}^{\perp}$, then (10) gives:

$$\int d^3r\ (\vec{E}^{\parallel})^* \cdot \vec{B}^{\perp} = 0; \tag{11}$$

we note that $\boldsymbol{\mathcal{E}}^*(\vec{k}) = \boldsymbol{\mathcal{E}}(-\vec{k})$ when $\vec{E}$ is a real field, in particular, from (11):

$$\int d^3r\ \vec{E}^{\parallel} \cdot \vec{E}^{\perp} = 0. \tag{12}$$

Similarly, with (10) we can obtain the expressions:

$$\int d^3r\ \vec{E}^{\parallel} \times \vec{B}^{\parallel} = \int d^3r\ \vec{E}^{\perp} \times \vec{B}^{\perp} = \vec{0}, \tag{13}$$

and the Rayleigh's energy theorem (1889):

$$\int d^3r\ (\vec{E}^{\parallel})^2 = \int d^3k\ |\boldsymbol{\mathcal{E}}^{\parallel}|^2, \qquad \int d^3r\ (\vec{E}^{\perp})^2 = \int d^3k\ |\boldsymbol{\mathcal{E}}^{\perp}|^2, \tag{14}$$

in fact, energy-preserving is an important property of Fourier transform.

The convolution integral introduced by Duhamel (1833) gives the useful relations:

$$\begin{aligned} \int d^3r'\ f(\vec{r}')\, g(\vec{r} - \vec{r}') &= \int d^3k\ e^{i\,\vec{k}\cdot\vec{r}}\, \mathcal{F}[f](\vec{k})\, \mathcal{F}[g](\vec{k}), \\ \int d^3r\ e^{-i\,\vec{k}\cdot\vec{r}}\, f(\vec{r})\, g(\vec{r}) &= \int d^3k'\ \mathcal{F}[f](\vec{k}')\, \mathcal{F}[g](\vec{k} - \vec{k}'), \end{aligned} \tag{15}$$

implying the integral representation for the Lanczos [11]-Dirac [12] delta:

$$\delta(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d^3k\ e^{i\,\vec{k}\cdot(\vec{r}-\vec{r}')} \tag{16}$$

besides:

$$\mathcal{F}[\delta(\vec{r} - \vec{r}')] = \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{-i\,\vec{k}\cdot\vec{r}'}. \tag{17}$$

3. FOURIER TRANSFORMS

For a point charge at $\vec{r}'$, the Maxwell equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ for the Coulomb field leads to $\vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}'}{4\pi\, |\vec{r} - \vec{r}'|^3} \right) = \delta(\vec{r} - \vec{r}')$, where we can apply (6) and (17) to obtain the Fourier transformation:

$$\mathcal{F}\left[\frac{\vec{r}-\vec{r}'}{4\pi\,|\vec{r}-\vec{r}'|^3}\right] = -\frac{i}{(2\pi)^{3/2}}\,\frac{\vec{k}}{k^2}\,e^{-i\,\vec{k}\cdot\vec{r}'}\,, \tag{18}$$

in particular:

$$\mathcal{F}\left[\frac{\vec{r}}{4\pi\,r^3}\right] = -\frac{i}{(2\pi)^{3/2}}\,\frac{\vec{k}}{k^2}\,, \tag{19}$$

and the corresponding Laplace equation $\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta(\vec{r}-\vec{r}')$ implies the expressions:

$$\mathcal{F}\left[\frac{1}{4\pi\,|\vec{r}-\vec{r}'|}\right] = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{k^2}\,e^{-i\,\vec{k}\cdot\vec{r}'}, \qquad \mathcal{F}\left[\frac{1}{4\pi\,r}\right] = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{k^2}\,, \tag{20}$$

because:

$$\mathcal{F}[\vec{\nabla}\phi] = i\,\vec{k}\,\mathcal{F}[\phi], \qquad \mathcal{F}[\nabla^2\phi] = -k^2\,\mathcal{F}[\phi], \tag{21}$$

for a scalar field ϕ.

The relations (18),…, (21) are useful to study the Maxwell equations [13], in fact, the spatial Fourier transformation of the electromagnetic field allows to see more clearly the actual independent degrees of freedom of the field [1].

4. LONGITUDINAL AND TRANSVERSAL δ-FUNCTION OF BELINFANTE

It is evident that in the reciprocal space:

$$\boldsymbol{\mathcal{E}}^{\parallel} = (\hat{k}\cdot\boldsymbol{\mathcal{E}})\hat{k} = \frac{1}{k^2}(\vec{k}\cdot\boldsymbol{\mathcal{E}})\,\vec{k} \qquad \therefore \qquad \mathcal{E}^{\parallel}_m = \frac{k_m}{k^2}\sum_j k_j\,\mathcal{E}_j, \tag{22.a}$$

$$\mathcal{E}^{\perp}_m = \mathcal{E}_m - \mathcal{E}^{\parallel}_m = \sum_j \left(\delta_{mj} - \frac{k_m\,k_j}{k^2}\right)\mathcal{E}_j, \tag{22.b}$$

where: δ_{mj} is the Kronecker delta; thus, in natural manner, Belinfante [14] introduced the transverse δ-function:

$$\delta^{\perp}_{mj}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\,\mathcal{F}^{-1}\left[\delta_{mj} - \frac{k_m\,k_j}{k^2}\right] = \frac{1}{(2\pi)^3}\int d^3k\left(\delta_{mj} - \frac{k_m\,k_j}{k^2}\right)e^{i\,\vec{k}\cdot\vec{r}}, \tag{23}$$

that is:

$$\mathcal{F}\left[\delta^{\perp}_{mj}(\vec{r})\right] = \frac{1}{(2\pi)^{3/2}}\left(\delta_{mj} - \frac{k_m\,k_j}{k^2}\right). \tag{24}$$

If we apply $\mathcal{F}^{-1}$ to (22.b) and we employ (15) and (24):

$$E_m^{\perp}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_j \int d^3k \; e^{i\,\vec{k}\cdot\vec{r}} \left(\delta_{mj} - \frac{k_m\, k_j}{k^2}\right) \mathcal{E}_j = \sum_j \int d^3k \; e^{i\,\vec{k}\cdot\vec{r}} \, \mathcal{F}[E_j]\, \mathcal{F}[\delta_{mj}^{\perp}],$$

$$= \sum_j \int d^3r' \;\; E_j(\vec{r}')\, \delta_{mj}^{\perp}(\vec{r} - \vec{r}'), \qquad (25)$$

hence the transverse δ-function allows to construct the components $E_m^{\perp}$ from the quantities E_j, and (25) tells us that it is possible via integration, therefore, the connection between $\vec{E}^{\perp}$ and $\vec{E}$ is non-local [1]. We can make more explicit the relation (25) if first we study (23):

$$\delta_{mj}^{\perp}(\vec{r}) \overset{(16)}{=} \delta_{mj}\, \delta(\vec{r}) + \frac{1}{(2\pi)^3} \frac{\partial^2}{\partial r_m\, \partial r_j} \int d^3k \, \frac{1}{k^2} \, e^{i\,\vec{k}\cdot\vec{r}} \overset{(20)}{=} \delta_{mj}\, \delta(\vec{r}) + \frac{1}{4\pi} \frac{\partial^2}{\partial r_m\, \partial r_j} \frac{1}{r},$$

therefore:

$$\delta_{mj}^{\perp}(\vec{r}) = \frac{2}{3}\, \delta_{mj}\, \delta(\vec{r}) - \frac{1}{4\pi\, r^3}\left(\delta_{mj} - \frac{3 r_m\, r_j}{r^2}\right), \qquad (26)$$

where we applied the following expression of Stewart [3] with $n = 1$:

$$\frac{1}{4\pi} \frac{\partial^2}{\partial r_m\, \partial r_j} \frac{1}{r^n} = -n \left[\frac{1}{3\, r^{n-1}}\, \delta_{mj}\, \delta(\vec{r}) + \frac{1}{4\pi\, r^{n+2}}\left(\delta_{mj} - \frac{(n+2)\, r_m\, r_j}{r^2}\right)\right]. \qquad (27)$$

Now we employ (26) into (25) to deduce the non-local relationship [1]:

$$E_m^{\perp}(\vec{r}) = \frac{2}{3}\, E_m(\vec{r}) - \int d^3r' \, \frac{E_m(\vec{r}')}{4\pi\, |\vec{r}-\vec{r}'|^3} + 3 \sum_j \int d^3r' \, \frac{(r_m - r'_m)\,(r_j - r'_j)}{4\pi\, |\vec{r}-\vec{r}'|^5}\, E_j(\vec{r}'), \qquad (28)$$

which shows that the Helmholtz's decomposition (8) is non-trivial; besides, from (22.b), (25) and (28):

$$E_m^{\parallel}(\vec{r}) = \frac{1}{3}\, E_m(\vec{r}) + \frac{1}{4\pi} \int d^3r' \, \frac{E_m(\vec{r}')}{|\vec{r}-\vec{r}'|^3} - \frac{3}{4\pi} \sum_j \int d^3r' \, \frac{(r_m - r'_m)\,(r_j - r'_j)}{|\vec{r}-\vec{r}'|^5}\, E_j(\vec{r}'), \qquad (29)$$

$$= \sum_j \int d^3r' \;\; E_j(\vec{r}')\, \delta_{mj}^{\parallel}(\vec{r} - \vec{r}'), \qquad (30)$$

with the presence of the Belinfante's longitudinal δ-function [14]:

$$\delta_{mj}^{\parallel}(\vec{r}) = \delta_{mj}\, \delta(\vec{r}) - \delta_{mj}^{\perp}(\vec{r}) \overset{(27)}{=} -\frac{1}{4\pi} \frac{\partial^2}{\partial r_m\, \partial r_j} \frac{1}{r},$$

$$\overset{(26)}{=} \frac{1}{3}\, \delta_{mj}\, \delta(\vec{r}) + \frac{1}{4\pi\, r^3}\left(\delta_{mj} - \frac{3\, r_m\, r_j}{r^2}\right) \overset{(20)}{=} \frac{1}{(2\pi)^3} \int d^3k \, \frac{k_m\, k_j}{k^2} \, e^{i\,\vec{k}\cdot\vec{r}}, \qquad (31)$$

therefore:

$$\mathcal{F}\left[\delta^{\parallel}_{mj}(\vec{r})\right] = \frac{1}{(2\pi)^{3/2}}\,\frac{k_m\,k_j}{k^2}, \qquad \mathcal{F}\left[\delta^{\perp}_{mj}(\vec{r}) + \delta^{\parallel}_{mj}(\vec{r})\right] = \frac{1}{(2\pi)^{3/2}}\,\delta_{mj}\,. \tag{32}$$

We can indicate an extension of (27), and several properties of the transverse and longitudinal δ-function [3]:

$$\frac{\partial^2}{\partial r_m\,\partial r_j}\,\frac{r_c}{r^n} = r_c\,\frac{\partial^2}{\partial r_m\,\partial r_j}\,\frac{1}{r} - \frac{n}{r^{n+2}}\left(r_j\delta_{mc} + r_m\delta_{jc}\right), \qquad \frac{\partial}{\partial r_c}\,\delta^{\parallel}_{mj} = \frac{\partial}{\partial r_m}\,\delta^{\parallel}_{cj}\,, \qquad \delta^{\parallel}_{mj} = \delta^{\parallel}_{jm}\,,$$

$$\delta^{\perp}_{mj} = \delta^{\perp}_{jm}\,, \qquad \sum_c \frac{\partial}{\partial r_c}\,\delta^{\parallel}_{cj}(\vec{r}-\vec{r}\,') = \frac{\partial}{\partial r_j}\,\delta(\vec{r}-\vec{r}\,'), \quad \sum_m \frac{\partial}{\partial r_m}\,\delta^{\perp}_{mj}(\vec{r}-\vec{r}\,') = 0, \tag{33}$$

$$\delta^{a}_{mj}(\vec{r}-\vec{r}\,') = \sum_c \int d^3 r''\;\delta^{a}_{mc}(\vec{r}-\vec{r}\,'')\,\delta^{a}_{cj}(\vec{r}\,''-\vec{r}\,'), \qquad a = \perp, \parallel\,.$$

5. CONCLUSIONS

The spatial Fourier transform is important to analyze the Maxwell fields and their connections with the 4-potential in the Riemann-Lorenz and Coulomb gauges [1, 7-10, 13, 15-18], and thus to determine the true degrees of freedom of the electromagnetic field. The Fourier transform was very important in the first versions of QED [19-22], especially in the radiation gauge because the Coulomb gauge preserves causality [23]. Fresnel (1821) [24] showed that it is possible explain the polarization only if the light is totally transversal [25, 26]; the absence of longitudinal components in electromagnetic waves is not fortuitous, but is a direct consequence of gauge symmetry [27, 28].

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