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Relativistic motion of classical charged particles in a uniform electromagnetic field

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ABSTRACT

Here we show the importance of a systematic method to generate any integer power of a matrix $\underset{\sim}{A}$ in 2, 3 and 4 dimensions. This is motivated by the integration of Frenet–Serret formulae for the case of constant curvatures, and by the relativistic motion of a point charge into a homogeneous electromagnetic field, because in such situations it is necessary to calculate $\exp(\gamma A)$ being γ a scalar. By these reasons, we believe that this work can be useful for people interested in linear algebra, differential geometry and electrodynamics.

Keywords**:** Lorentz equation, Frenet-Serret formulae, Helixs in Minkowski spacetime

1. INTRODUCTION

We consider a path Γ into Euclidean plane with s denoting its arc length, then its intrinsic geometry is characterized by the curvature $k(s)$, which governs the rotation of the unitary vectors tangent $\vec{T}$ and nomal $\vec{N}$ to Γ, via the Frenet–Serret equations [1–9]:

$$\frac{d\vec{T}}{ds} = k\,\vec{N}\,, \qquad \frac{d\vec{N}}{ds} = -k\,\vec{T} \tag{1}$$

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where: $\vec{T} = \frac{d\vec{r}}{ds}$ being $\vec{r}(s)$ the position vector. With the matrices:

$$\underset{\sim}{M} = \begin{pmatrix} T_1 & T_2 \\ N_1 & N_2 \end{pmatrix} = \begin{pmatrix} \vec{T} \\ \vec{N} \end{pmatrix}, \qquad \underset{\sim}{k} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \tag{2}$$

it is clear that (1) can be written in the form:

$$\frac{d}{ds}\underset{\sim}{M} = \underset{\sim}{k}\,\underset{\sim}{M}\,. \tag{3}$$

If now we want to know the trajectories with $k = constant$, we can resolve –as is usual in books of differential geometry– the following equation from (1):

$$\frac{d^2\vec{T}}{ds^2} + k^2\vec{T} = \vec{0}\,,$$

or, to integrate (3) with the initial orientation $(\vec{T}_0, \vec{N}_0)$:

$$\underset{\sim}{M}(s) = \exp\left(s\,\underset{\sim}{k}\right)\underset{\sim}{M}(0), \tag{4}$$

appearing thus the exponential function of a matrix. In this case we have $k_{ab} = -k_{ba}$, however, the $\exp\left(s\,\underset{\sim}{k}\right)$ can be calculated exactly without the antisymmetry of $\underset{\sim}{k}$. From (4) results that the circumference is the only plane path of constant curvature, and it is generated by:

$$\vec{r}(s) = \vec{r}(0) + \int_0^s \vec{T}(\theta)\, d\theta\,. \tag{5}$$

Besides, as $(\vec{T}, \vec{N})$ differs of $(\vec{T}_0, \vec{N}_0)$ by a rotation, then $\exp(s\,\underset{\sim}{k})$ is an orthogonal matrix (its multiplicaction with its transpose gives the identity). The expansion –for $\underset{\sim}{k}$ arbitrary–:

$$\exp\left(s\,\underset{\sim}{k}\right) = \sum_{r=0}^{\infty} \frac{s^r}{r!}\underset{\sim}{k}^r\,, \tag{6}$$

reduces our problem to calculation of powers $\underset{\sim}{k}^r$, which it was realized by Synge [10] for the case 2 × 2. In the literature we have not found the corresponding study for 3 and 4 dimensions. The expressions (6) also are interesting when $\underset{\sim}{k}$ coincides with a matrix of Pauli, with application thus to angular momentum in quantum mechanics and in classical mechanics to rigid body dynamics [11]. A similar investigation can be realized when Γ is embedded into Euclidean 3–space, only we need the torsion $\tau(s)$ with the unitary binormal $\vec{B}(s)$ and the corresponding Frenet–Serret formulae:

$$\frac{d\vec{T}}{ds} = k\vec{N}, \qquad \frac{d\vec{N}}{ds} = -k\vec{T} + \tau\,\vec{B}, \qquad \frac{d\vec{B}}{ds} = -\tau\,\vec{N}, \tag{7}$$

with the matricial representation (3) and the definitions:

$$\underset{\sim}{M} = \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}, \qquad \underset{\sim}{k} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \tag{8}$$

If we again desire that Γ has constant torsion and curvature constants, then we can resolve the following differential equation deduced from (7):

$$\frac{d^3}{ds^3}\vec{T} + \left(k^2 + \tau^2\right)\frac{d\vec{T}}{ds} = \bar{0} \ ,$$

or in (4) to make explicit the exponential function of $s\underset{\sim}{k}$ and thus to prove that Γ is a circular helix. This leads to determination of $\underset{\sim}{k}^r$, $r = 0,1,2,\ldots,$ which motivates to know the eigenvalues of $\underset{\sim}{k}$, its 3th order characteristic polynomial admits solution via Ferrari-Tartaglia formulae. If the concept of helix is generalized to $\frac{k}{\tau} = b = constant$, then we can multiply (7) by $\frac{1}{\tau}$ and to introduce the parameter $\beta = \int_0^s \tau(\theta)\, d\theta$ for to obtain the relations:

$$\frac{d\vec{T}}{d\beta} = b\vec{N}, \qquad \frac{d\vec{N}}{d\beta} = -b\vec{T} + \vec{B}, \qquad \frac{d\vec{B}}{d\beta} = -\vec{N},$$

whose integration is immediate if in the solution of (7) we make the changes $s \to \beta$, $k \to b$, $\tau \to 1$.

2. LORENTZ EQUATION IN MINKOWSKI SPACE

Now we consider the motion –in the spacetime– of a point charge q under the action of a constant electromagnetic field characterized by the Faraday's antisymmetric matrix:

$$\underset{\sim}{F} = \left(F_{jr}\right) = \begin{pmatrix} 0 & B_z & -B_\gamma & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}, \quad i = \sqrt{-1} \tag{9}$$

where: $\vec{E}$ and $\vec{B}$ are the electric and magnetic vectors. Then the interaction of q with the external field is controlled by the Lorentz equation [12]:

$$\frac{d}{ds} V_r = a F_{rc} V_c , \qquad a = \frac{q}{m_0}, \tag{10}$$

being m_0, s and V_r the rest mass, proper time and 4–velocity of the particle, respectively, accepting the notation of [1,12,13] and the convention of sum over repeated indices.

The problem is to resolve (10) for $\underset{\sim}{F}$ constant, which was realized independently by Synge [1], Honig et al [3], Plebañski [14] and Piña [15], however, their methods have not compact formulae for $\underset{\sim}{F}^{r}$, in terms of the eigenvalues of $\underset{\sim}{F}$, and this is necessary because the solution of (10) is given by:

$$V_r(s) = \exp(asF_{rc}) V_c(0), \tag{11}$$

with $V_c(0)$ denoting the initial velocity of the charge. If we know $V_r(s)$ then the trajectory can be determined from:

$$x_r(s) = x_r(0) + \int_0^s V_r(\theta) d\theta , \tag{12}$$

and thus the study of the relativistic motion is reduced to computation of the exponential function of matrix $F_{jc} = -F_{cj}$, which makes necessary to have an expression for the integer powers of $\underset{\sim}{F}$. For this reason, in the case 4 × 4 we limit us to antisymmetric matrices.

3. OUR METHOD TO DETERMINE $\exp(\gamma \underset{\sim}{Q})$.

As was mentioned, here only we study $n \times n$ matrices with n = 2, 3, 4, however, our method has application to any dimension; then we shall explain it for the general case indicating its principal aspects:

a).- To determine the characteristic polynomial of $\underset{\sim}{Q}$:

$$P(\lambda) = \det\left(\underset{\sim}{Q} - \lambda \underset{\sim}{I}\right), \tag{13}$$

where: $\underset{\sim}{I}$ is the $n \times n$ identity matrix; there are efficient techniques [16-21] to construct $P(\lambda)$ for arbitrary n.

b).- To resolve the characteristic equation:

$$P(\lambda) = 0 . \tag{14}$$

In general it is difficult to find exact solutions of (14), but we have algorithms [22-25] for to deduce the corresponding approximated roots.

c).- To obtain the coefficients q_j in the expansion:

$$\underset{\sim}{Q}^r = q_n \underset{\sim}{I} + q_{n-1} \underset{\sim}{Q} + \cdots + q_1 \underset{\sim}{Q}^{n-1}, \qquad r = n+1, n+2, \ldots \tag{15}$$

when the eigenvalues λ_j, $j = 1, \ldots, n$, are differents, which is equivalent to resolve the system:

$$\begin{array}{c} \lambda_1^{n-1} q_1 + \lambda_1^{n-2} q_2 + \cdots + \lambda_1 q_{n-1} + q_n = \lambda_1^r , \\ \vdots \qquad\qquad\qquad \vdots \\ \lambda_n^{n-1} q_1 + \lambda_n^{n-2} q_2 + \cdots + \lambda_n q_{n-1} + q_n = \lambda_n^r , \end{array} \tag{16}$$

which results if we multiply (5) by each one eigenvectors. If some proper values are identical, for example $\lambda_1 = \lambda_2$, then the corresponding q_j are obtained with the $\lim\limits_{\lambda_2 \to \lambda_1} q_j(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We note that in (15) were excluded $r = 0, 1, \ldots, n,$ that is, if $n = 4$ then is necessary to calculate directly $\underset{\sim}{Q}^2, \underset{\sim}{Q}^3$ and $\underset{\sim}{Q}^4$, we remember that the Cayley–Hamilton theorem [18-20] is useful for to determine $\underset{\sim}{Q}^4$.

d).- To substitute $q_j(\lambda_1, \ldots, \lambda_n)$ into (15) and to obtain explicitly:

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \sum_{r=0}^{\infty} \frac{\gamma^r}{r!} \underset{\sim}{Q}^r \ . \tag{17}$$

4. LEVERRIER-TAKENO ALGORITHM FOR $P(\lambda)$.

For arbitrary $\underset{\sim}{Q}_{n\times n}$ its $P(\lambda)$ is given by [16, 17, 20, 26-30] the expansion polynomial:

$$P(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n \,, \tag{18}$$

with the recurrence relation:

$$p\alpha_p + s_1\alpha_{p-1} + s_2\alpha_{p-2} + \cdots + s_{p-1}\alpha_1 + s_p = 0, \quad p = 1,\ldots,n$$
$$(19)$$
$$\alpha_0 = 1, \quad s_j = spur\,\underset{\sim}{Q}^j = \lambda_1^j + \cdots + \lambda_n^j, \quad j = 1,\ldots,n.$$

For example:

$$\alpha_1 = -s_1, \quad \alpha_2 = \frac{1}{2}\left(s_1^2 - s_2\right), \quad \alpha_3 = \frac{1}{6}\left(-s_1^3 + 3s_1 s_2 - 2s_3\right),$$
$$(20)$$
$$\alpha_4 = \frac{1}{24}\left(s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4\right), \ldots, \ \alpha_n = (-1)^n \det \underset{\sim}{Q}.$$

Besides, it is clear that when $\underset{\sim}{Q}$ is antisymmetric then $s_j = \alpha_j = 0$ for j odd. Now we consider three cases of interest:

i) $n = 2$ **.**

$$P(\lambda) = \lambda^2 + \alpha_1\lambda + \alpha_2 = \lambda^2 - s_1\lambda + \frac{1}{2}\left(s_1^2 - s_2\right) \quad , \tag{21}$$

and if $Q_{jc} = -Q_{cj}$ then:

$$P(\lambda) = \lambda^2 - \frac{1}{2} s_2 = \lambda^2 + \left(Q_{12}\right)^2 , \tag{22}$$

therefore its eigenvalues are pure imaginaries, this it will occur with the curvature matrix (2). From (22) is evident that $P(0) = \det \underset{\sim}{Q} = (Q_{12})^2$, which is a particular case of the theorem [31, 32]:

"Det $\underset{\sim}{Q}$ for antisymmetric $Q_{n\times n}$ with n even, is the square of a rational polynomial in Q_{jc}" (23)

ii) $n = 3$.

$$P(\lambda) = \lambda^3 - s_1\lambda^2 + \frac{1}{2}\left(s_1^2 - s_2\right)\lambda + \frac{1}{6}\left(-s_1^3 + 3s_1s_2 - 2s_3\right), \tag{24}$$

and when $\underset{\sim}{Q}$ is antisymmetric:

$$P(\lambda) = \lambda\left(\lambda^2 + Q_{12}^2 + Q_{13}^2 + Q_{23}^2\right), \tag{25}$$

thus one proper value is zero and the others are imaginaries.

iii) $n = 4$ and $Q_{ab} = -Q_{ba}$.

$$P(\lambda) = \lambda^4 + \left(Q_{12}^2 + Q_{13}^2 + Q_{14}^2 + Q_{23}^2 + Q_{24}^2 + Q_{34}^2\right)\lambda^2 + \det \underset{\sim}{Q}, \tag{26}$$

such that:

$$\det \underset{\sim}{Q} = \left(Q_{12}Q_{34} + Q_{13}Q_{24} + Q_{23}Q_{14}\right)^2, \tag{27}$$

which is one more example of (23).

In (26) we see that if λ is an eigenvalue, then $-\lambda$ also is, therefore the proper values appear as pairs $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$, with great simplification for the case 4 × 4.

5. POWERS OF A MATRIX

Similary to Sec. 4, here we shall study three situations:

I) $\underset{\sim}{Q}$ is 2 × 2.

Then (15) adopts the form:

$$\underset{\sim}{Q}^r = q_2 \underset{\sim}{I} + q_1 \underset{\sim}{Q} \ , \qquad r = 3,4,... \tag{28}$$

and when $\lambda_1 \neq \lambda_2$ from (16) we obtain the system:

$$\lambda_1 q_1 + q_2 = \lambda_1^r , \qquad \lambda_2 q_1 + q_2 = \lambda_2^r \ ,$$

whose solution is immediate and it coincides with the Synge relation [10]:

$$q_1 = \frac{\lambda_1^r - \lambda_2^r}{\lambda_1 - \lambda_2} , \qquad q_2 = \frac{\lambda_1 \lambda_2^r - \lambda_1^r \lambda_2}{\lambda_1 - \lambda_2} \ , \tag{29}$$

besides, (17), (28) and (29) imply the Hildebrand expression [33]:

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \frac{1}{\lambda_1 - \lambda_2}\left[\left(\lambda_1 e^{\gamma\lambda_2} - \lambda_2 e^{\gamma\lambda_1}\right)\underset{\sim}{I} + \left(e^{\gamma\lambda_1} - e^{\gamma\lambda_2}\right)\underset{\sim}{Q}\right]. \tag{30}$$

Thus we see that the 2 × 2 case is resolved in the literature, then our contribution corresponds to $n = 3,4$. When $\lambda_1 = \lambda_2$, then the application into (29) and (30) of Hôpital rule for the $\underset{\lambda_2 \to \lambda_1}{lím}$ leads to relations:

$$q_1 = r\lambda_1^{r-1} , \qquad q_2 = (r-1)\lambda_1^r , \qquad \exp\left(\gamma \underset{\sim}{Q}\right) = \left[(1 - \gamma\lambda_1)\underset{\sim}{I} + \gamma \underset{\sim}{Q}\right] e^{\gamma\lambda_1} . \tag{31}$$

II) $\underset{\sim}{Q}$ is 3 × 3.

From (15) and (16) we deduce that for $r = 4,5,...$:

$$\underset{\sim}{Q}^r = q_3 \underset{\sim}{I} + q_2 \underset{\sim}{Q} + q_1 \underset{\sim}{Q}^2 , \quad \lambda_1^2 q_1 + \lambda_1 q_2 + q_3 = \lambda_1^r ,$$

$$\lambda_2^2 q_1 + \lambda_2 q_2 + q_3 = \lambda_2^r \ , \quad \lambda_3^2 q_1 + \lambda_3 q_2 + q_3 = \lambda_3^r ,$$

with solution –for different proper values– given by:

$$q_1 = \frac{1}{D}\left[\lambda_1^r(\lambda_2 - \lambda_3) + \lambda_2^r(\lambda_3 - \lambda_1) + \lambda_3^r(\lambda_1 - \lambda_2)\right],$$

$$q_2 = \frac{1}{D}\left[\lambda_1^r(\lambda_3^2 - \lambda_2^2) + \lambda_2^r(\lambda_1^2 - \lambda_3^2) + \lambda_3^r(\lambda_2^2 - \lambda_1^2)\right], \tag{32}$$

$$q_3 = \frac{1}{D}\left[\lambda_1^r \lambda_2 \lambda_3(\lambda_2 - \lambda_3) + \lambda_1 \lambda_2^r \lambda_3(\lambda_3 - \lambda_1) + \lambda_1 \lambda_2 \lambda_3^r(\lambda_1 - \lambda_2)\right],$$

$$D = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3),$$

resulting thus the following structure for (17):

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \frac{1}{D}\Big\{\left[\lambda_2\lambda_3(\lambda_2 - \lambda_3)e^{\gamma\lambda_1} + \lambda_1\lambda_3(\lambda_3 - \lambda_1)e^{\gamma\lambda_2} + \lambda_1\lambda_2(\lambda_1 - \lambda_2)e^{\gamma\lambda_3}\right]\underset{\sim}{I} +$$
$$+\left[(\lambda_3^2 - \lambda_2^2)e^{\gamma\lambda_1} + (\lambda_1^2 - \lambda_3^2)e^{\gamma\lambda_2} + (\lambda_2^2 - \lambda_1^2)e^{\gamma\lambda_3}\right]\underset{\sim}{Q} + \qquad (33)$$
$$+\left[(\lambda_2 - \lambda_3)e^{\gamma\lambda_1} + (\lambda_3 - \lambda_1)e^{\gamma\lambda_2} + (\lambda_1 - \lambda_2)e^{\gamma\lambda_3}\right]\underset{\sim}{Q}^2\Big\}.$$

If $\lambda_1 = \lambda_2 \neq \lambda_3$, then applying the Hôpital rule to (32) and (33) we deduce:

$$q_1 = \frac{1}{\bar{D}}\left[(r-1)\lambda_1^r - r\lambda_1^{r-1}\lambda_3 + \lambda_3^r\right], \quad \bar{D} = (\lambda_1 - \lambda_3)^2,$$

$$q_2 = \frac{1}{\bar{D}}\left[(2-r)\lambda_1^{r-1} + r\lambda_1^{r-1}\lambda_3^2 - 2\lambda_1\lambda_3^r\right],$$

$$q_3 = \frac{1}{\bar{D}}\left[(r-2)\lambda_1^{r+1}\lambda_3 + (1-r)\lambda_1^r\lambda_3^2 + \lambda_1^2\lambda_3^r\right], \qquad (34)$$

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \frac{1}{\bar{D}}\Big\{\left[\lambda_3(\gamma\lambda_1^2 - \gamma\lambda_1\lambda_3 - 2\lambda_1 + \lambda_3)e^{\gamma\lambda_1} + \lambda_1^2 e^{\gamma\lambda_3}\right]\underset{\sim}{I}$$
$$+\left[(-\gamma\lambda_1^2 + 2\lambda_1 + \gamma\lambda_3^2)e^{\gamma\lambda_1} - 2\lambda_1 e^{\gamma\lambda_3}\right]\underset{\sim}{Q} + \left[(\gamma\lambda_1 - \gamma\lambda_3 - 1)e^{\gamma\lambda_1} + e^{\gamma\lambda_3}\right]\underset{\sim}{Q}^2\Big\}.$$

Finally, if $\lambda_1 = \lambda_2 = \lambda_3$ from (34) we obtain:

$$q_1 = \frac{r}{2}(r-1)\lambda_1^{r-2}, \quad q_2 = r(2-r)\lambda_1^{r-1}, \quad q_3 = \frac{1}{2}(r-1)(r-2)\lambda_1^r,$$

$$(35)$$

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \frac{1}{2}\left[(\gamma^2\lambda_1^2 - 2\gamma\lambda_1 + 2)\underset{\sim}{I} + 2\gamma(1 - \gamma\lambda_1)\underset{\sim}{Q} + \gamma^2\underset{\sim}{Q}^2\right]e^{\gamma\lambda_1}.$$

III) $\underset{\sim}{Q}_{4\times 4}$ is antisymmetric.

In this case (15) and (16) lead to:

$$\underset{\sim}{Q}^r = q_4\,\underset{\sim}{I} + q_3\,\underset{\sim}{Q} + q_2\,\underset{\sim}{Q}^2 + q_1\,\underset{\sim}{Q}^3, \qquad r = 5,6,...$$

$$\lambda_1^3 q_1 + \lambda_1^2 q_2 + \lambda_1 q_3 + q_4 = \lambda_1^r, \qquad -\lambda_1^3 q_1 + \lambda_1^2 q_2 - \lambda_1 q_3 + q_4 = (-1)^r \lambda_1^r,$$

$$\lambda_3^3 q_1 + \lambda_3^2 q_2 + \lambda_3 q_3 + q_4 = \lambda_3^r, \qquad -\lambda_3^3 q_1 + \lambda_3^2 q_2 - \lambda_3 q_3 + q_4 = (-1)^r \lambda_3^r,$$

which imply the following relations with $D' = \left[2\left(\lambda_1^2 - \lambda_1^3\right)\right]^{-1}$:

$$q_1 = D'\left(1-(-1)^r\right)\left(\lambda_1^{r-1} - \lambda_3^{r-1}\right), \quad q_2 = D'\left(1+(-1)^r\right)\left(\lambda_1^r - \lambda_3^r\right),$$

$$q_3 = D'\left(1-(-1)^r\right)\left(\lambda_1^2\lambda_3^{r-1} - \lambda_1^{r-1}\lambda_3^2\right), \quad q_4 = D'\left(1+(-1)^r\right)\left(\lambda_1^2\lambda_3^r - \lambda_1^r\lambda_3^2\right),$$

(36)

$$\exp\left(\gamma \underset{\sim}{Q}\right) = 2D'\Big[\left(\lambda_1^2 Cosh(\gamma\lambda_3) - \lambda_3^2 Cosh(\gamma\lambda_1)\right)\underset{\sim}{I} + \left(\lambda_1^2\lambda_3^{-1} Sinh(\gamma\lambda_3) - \lambda_1^{-1}\lambda_3^2 Sinh(\gamma\lambda_1)\right)\underset{\sim}{Q}$$

$$+\left(Cosh(\gamma\lambda_1) - Cosh(\gamma\lambda_3)\right)\underset{\sim}{Q}^2 + \left(\lambda_1^{-1} Sinh(\gamma\lambda_1) - \lambda_3^{-1} Sinh(\gamma\lambda_3)\right)\underset{\sim}{Q}^3\Big].$$

when: $\lambda_1 = \lambda_3 \neq 0$ these expressions (36) adopt the form:

$$q_1 = \frac{1}{4}\left(1-(-1)^r\right)(r-1)\lambda_1^{r-3}, \qquad q_2 = \frac{1}{4}\left(1+(-1)^r\right) r\lambda_1^{r-2},$$

$$q_3 = -\frac{1}{4}\left(1-(-1)^r\right)(r-3)\lambda_1^{r-1}, \qquad q_4 = -\frac{1}{4}\left(1+(-1)^r\right)(r-2)\lambda_1^r,$$

(37)

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \frac{1}{2}\Big[\left(2Cosh(\gamma\lambda_1) - \gamma\lambda_1 Sinh(\gamma\lambda_1)\right)\underset{\sim}{I} + \left(3\lambda_1^{-1} Sinh(\gamma\lambda_1) - \gamma Cosh(\gamma\lambda_1)\right)\underset{\sim}{Q}$$

$$+\gamma\lambda_1^{-1} Sinh(\gamma\lambda_1)\underset{\sim}{Q}^2 + \lambda_1^{-2}\left(\gamma Cosh(\gamma\lambda_1) - \lambda_1^{-1} Sinh(\gamma\lambda_1)\right)\underset{\sim}{Q}^3\Big],$$

and if $\lambda_1 = \lambda_3 = 0$ is evident that:

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \underset{\sim}{I} + \gamma \underset{\sim}{Q} + \frac{1}{2}\gamma^2 \underset{\sim}{Q}^2, \qquad (38)$$

because for such case it can be showed that $\underset{\sim}{Q}^3 = \underset{\sim}{0}$.

Finally, if $\lambda_1 \neq \lambda_3 = 0$:

$$q_1 = \frac{1}{2}\left(1-(-1)^r\right)\lambda_1^{r-3}, \qquad q_2 = \frac{1}{2}\left(1+(-1)^r\right)\lambda_1^{r-2}, \qquad q_3 = q_4 = 0,$$

(39)

$$\exp\left(\gamma \underset{\sim}{Q}\right) = \underset{\sim}{I} + \gamma \underset{\sim}{Q} + \lambda_1^{-2}\left(Cosh(\gamma\lambda_1)-1\right)\underset{\sim}{Q}^2 + \lambda_1^{-2}\left(\lambda_1^{-1}Sinh(\gamma\lambda_1)-\gamma\right)\underset{\sim}{Q}^3 .$$

6. LORENTZ AND FRENET-SERRET EQUATIONS

Here we shall apply the results of Sec. 5 to computation of trajectories with constant curvatures in 2, 3 and 4 dimensions. Synge [1] proved that the worldline of a charge –under the action of a homogeneous electromagnetic field– has constant curvatures, that is, it represents a helix in Minkowski spacetime.

We consider an Euclidean plane with $\underset{\sim}{Q} = \underset{\sim}{k}$ given by (2) and $\gamma = s$, then $\lambda_1 = -\lambda_2 = ik$ and thus (4), (5), (30) imply:

$$\exp\left(s\underset{\sim}{k}\right) = \begin{pmatrix} Cos(ks) & Sin(ks) \\ -Sin(ks) & Cos(ks) \end{pmatrix}, \qquad \vec{T}(s) = Cos(ks)\vec{T}(0) + Sin(ks)\vec{N}_0,$$

(40)

$$\vec{r}(s) = \vec{r}(0) + \frac{1}{k}Sin(ks)\vec{T}_0 + \frac{1}{k}\left(1-Cos(ks)\right)\vec{N}_0,$$

which means a circumference of radius $\frac{1}{k}$ with center at $\vec{r}_0 + \frac{1}{k}\vec{N}_o$.

In 3 dimensions we have $\underset{\sim}{Q} = \underset{\sim}{k}$ indicated in (8) being $\gamma = s$; from (25) is clear that $\lambda_1 = 0$, $\lambda_2 = -\lambda_3 = iH$, $H = \sqrt{k^2+\tau^2}$, then (4), (5) and (33) lead to:

$$\exp\left(s\underset{\sim}{k}\right) = \underset{\sim}{I} + \frac{1}{H}Sin(Hs)\underset{\sim}{k} + \frac{1}{H^2}\left(1-Cos(Hs)\right)\underset{\sim}{k}^2, \qquad (41)$$

$$\vec{r}(s)=\vec{r}(0)+\left[s-\frac{k^2}{H^2}\left(s-\frac{1}{H}Sin(Hs)\right)\right]\vec{T}_0+\frac{k}{H^2}(1-Cos(Hs))\vec{N}_0+\frac{k\tau}{H^2}\left(s-\frac{1}{H}Sin(Hs)\right)\vec{B}_0,$$

that represents a circular helix of step $\frac{2\pi\tau}{H^2}$, radius $\frac{k}{H^2}$ and axis $\vec{L}_0=\tau\vec{T}_0+k\vec{B}_0$.

Ultimately, we suppose that $\underset{\sim}{Q}$ is equal to Faraday matrix given by (9) and $\gamma=as$. Therefore, from (26) and (27) we deduce the characteristic equation:

$$P(\lambda)=\lambda^4+\frac{1}{2}I_1\lambda^2-\frac{1}{16}I_2^2=0\ , \qquad (42)$$

being I_1 and I_2 the Lorentz invariants:

$$I_1=2\left(B^2-E^2\right),\quad I_2=4\vec{E}\cdot\vec{B},\quad E=\left|\vec{E}\right|,\quad B=\left|\vec{B}\right|, \qquad (43)$$

which permit to introduce the Synge-Piña classification [1, 5, 15, 32] for the electromagnetic field:

$$\begin{aligned}&\text{Type A: } I_2\neq 0,\qquad \text{Type B: } I_1<0,\qquad I_2=0,\\ &\text{Type C: } I_1=I_2=0,\qquad \text{Type D: } I_1>0,\qquad I_2=0;\end{aligned} \qquad (44)$$

the type C also is named Null Field. Then we shall consider the following four cases:

Type A:

$$\lambda_1^2=\frac{1}{4}\left(-I_1+\sqrt{I_1^2+I_2^2}\right),\qquad \lambda_3^2=-\lambda^2=-\frac{1}{4}\left(I_1+\sqrt{I_1^2+I_2^2}\right),$$

than we can substitute in (36) to obtain:

$$\begin{aligned}\exp\left(as\underset{\sim}{F}\right)=\frac{2}{\sqrt{I_1^2+I_2^2}}\Big[&\left(\lambda_1^2Cos(as\lambda)+\lambda^2Cosh(as\lambda_1)\right)\underset{\sim}{I}+\left(\lambda_1^2\lambda^{-1}Sin(as\lambda)+\lambda_1^{-1}\lambda^2Sinh(as\lambda_1)\right)\underset{\sim}{F}+\\ &+\left(Cosh(as\lambda_1)-Cos(as\lambda)\right)\underset{\sim}{F}^2+\left(\lambda_1^{-1}Sinh(as\lambda_1)-\lambda^{-1}Sin(as\lambda)\right)\underset{\sim}{F}^3\Big]\ .\end{aligned} \qquad (45)$$

Type B:

Here we have $\lambda_1 = \left(-\frac{I_1}{2}\right)^{1/2}$ and $\lambda_3 = 0$, then (39) generates the expansion:

$$\exp\left(as\underset{\sim}{F}\right) = \underset{\sim}{I} + as\underset{\sim}{F} + \frac{2}{I_1}\left(11 - Cosh(as\lambda_1) - 1\right)\underset{\sim}{F}^2 + \frac{2}{I_1}\left(as - \lambda_1^{-1}Sinh(as\lambda_1)\right)\underset{\sim}{F}^3 . \qquad (46)$$

Type C:

The null field implies $\lambda_1 = \lambda_3 = 0$ then from (38) results the relations:

$$\exp\left(as\underset{\sim}{F}\right) = \underset{\sim}{I} + as\underset{\sim}{F} + \frac{1}{2}a^2 s^2 \underset{\sim}{F}^2 \qquad (47)$$

Type D:

This case corresponds to $\lambda_1 = i\lambda = i\left(\frac{I_1}{2}\right)^{1/2}$ and $\lambda_3 = 0$, thus (39) leads to:

$$\exp\left(as\underset{\sim}{F}\right) = \underset{\sim}{I} + as\underset{\sim}{F} + \frac{2}{I_1}\left(1 - Cos(as\lambda)\right)\underset{\sim}{F}^2 + \frac{2}{I_1}\left(as - \lambda^{-1}Sin(as\lambda)\right)\underset{\sim}{F}^3 . \qquad (48)$$

If we put (45) ,..., (48) into (11) and (12) we obtain the paths of the particle, in complete harmony with the expressions of [1, 3, 14, 15, 34-39]; therefore, the motion of the charge is known in exact manner.

7. CONCLUSIONS

Our work shows the importance of the exponential function of a matrix to study helixs and to determine the relativistic motion of point charges into a homogeneous electromagnetic field. In the present analysis are fundamental the Leverrier-Takeno method to obtain the characteristic equation for the eigenvalues, and the Cayley-Hamilton theorem.

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