Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 96 (2018) 96-107 EISSN 2392-2192

Riesz Triple Probabilisitic of Almost Lacunary Cesàro C_{111} Statistical Convergence of Γ^3 Defined by Musielak Orlicz Function

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ABSTRACT

In this paper we study the concept of almost lacunary statistical Cesa'ro of Γ^3 over probabilistic p- metric spaces defined by Musielak Orlicz function. Since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis, we feel that the concept of almost lacunary statistical Cesàro of Γ^3 over probabilistic p- metric spaces defined by Musielak-Orlicz function in a PP-space would provide a more general framework for the subject.

Keywords**:** Analytic sequence, Orlicz function, triple sequences, entire sequence, Riesz space, statistical convergence

2010 Mathematics Subject Classification: 40F05, 40J05, 40G05

(Received 16 March 2018; Accepted 30 March 2018; Date of Publication 31 March 2018)

1. INTRODUCTION

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [2], Deepmala et al. [8,9] and many others. The initial work on triple sequence spaces is found in Sahiner et al. [10], Esi [3] and Esi et al. [4-7], Subramanian et al. [11], Shri Prakash et al. [12] and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. Then the triple series is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1,2,3, \dots) .$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequence is usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

A sequence $x = (x_{mnk})$ is called triple entire sequence if $(|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple entire sequences will be denoted by Γ^3.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & \dots 0 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \dots 1 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \end{bmatrix}$$

with 1 in the $(i, j, q)^{th}$ position and zero otherwise. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [14] as follows

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where: $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

Consider the triple difference sequence space is defined as

$$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$$

and

$$\Delta^0 x_{mnk} = \langle x_{mnk} \rangle.$$

2. DEFINITIONS AND PRELIMINARIES

2. 1. Definition

An Orlicz function ([see [13]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([15]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1,2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [16]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \quad as \quad m, n, k \to \infty\right\},$$

where: I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

2. 2. Definition

A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to limit 0 if

$$P - lim_{p,q,u \to \infty} sup_{r,s,t \geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k} = 0.$$

That is, the average value of (x_{mnk}) taken over any rectangle $\{(m,n,k): r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ tends to 0 as both p, q and u to ∞, and this $P-$ convergence is uniform in i, ℓ and j. Let denote the set of sequences with this property as $[\widehat{\chi^3}]$.

2. 3. Definition

Let $(Q_r), (\overline{Q_s}), (\overline{\overline{Q_t}})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1s} & 0\dots \\ q_{21} & q_{22} & \dots & q_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \dots & q_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = q_{11} + q_{12} + \cdots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \dots & \overline{q}_{1s} & 0\dots \\ \overline{q}_{21} & \overline{q}_{22} & \dots & \overline{q}_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \dots & \overline{q}_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \dots & \overline{\overline{q}}_{1s} & 0\dots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \dots & \overline{\overline{q}}_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \dots & \overline{\overline{q}}_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \neq 0$$

and is given by:

$$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k |x_{mnk}|^{1/m+n+k}$$

is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0{,}0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.

2. 4. Definition

The four dimensional matrix A is said to be RH-regular if it maps every bounded $P -$ convergent sequence into a $P -$ convergent sequence with the same $P -$ limit.

2. 5. Definition

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k): m_{i-1} < m < m_i \quad and \quad n_{\ell-1} < n \le n_\ell \quad and \quad k_{j-1} < k \le k_j\},$$
$$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary Fuzzy sequence and Riesz mean for triple sequences.

$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m \in (0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0,k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0,m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$.

If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.

Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \cdots + q_{rs} \to \infty (r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty (s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \to \infty (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.

Let $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j, I'_{i\ell j} = \left\{(m,n,k): Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}, V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i\overline{V}_\ell\overline{\overline{V}}_j$.

If we take $q_m = 1, \overline{q}_n = 1$ *and* $\overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \dots, x_n) = \| (d_1(x_1), \dots, d_n(x_n)) \|_p$ on X satisfying the following four conditions:

(i). $\| (d_1(x_1), \dots, d_n(x_n)) \|_p = 0$ if and and only if $d_1(x_1), \dots, d_n(x_n)$ are linearly dependent,

(ii). $\| (d_1(x_1), \dots, d_n(x_n)) \|_p$ is invariant under permutation,

(iii). $\| (\alpha d_1(x_1), \dots, \alpha d_n(x_n)) \|_p = |\alpha| \, \| (d_1(x_1), \dots, d_n(x_n)) \|_p, \alpha \in \mathbb{R}$

(iv). $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} for 1 \leq p < \infty$; is called the p product metric.

3. ALMOST LACUNARY CESÀRO C_{111} − STATISTICAL CONVERGENCE OF PP TRIPLE SEQUENCE SPACES

Let $A = \left[a_{mnk}^{pqr}\right]_{m,n,k=0}^{\infty}$ be a triple infinite matrix of real number for $p, q, r = 1, 2, \cdots$ forming the sum

$$\mu_{pqr}(X) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{mnk}^{pqr} \left(\left(\frac{X_{mnk}}{Y_{mnk}} \right)^{1/m+n+k}, 0 \right) \tag{3.1}$$

is called a triple sequence space of summable to the limit 0, i.e.,

$$lim_{uvw \to \infty} \sum_m^u \sum_n^v \sum_k^w a_{mnk}^{pqr} \left(\left(\frac{X_{mnk}}{Y_{mnk}} \right) \right)^{1/m+n+k} = \mu_{pqr}$$

and

$$\lim_{pqr\to\infty}\mu_{pqr} = 0$$

Define the means

$$\sigma_{pqr}^{X} = \frac{1}{pqr}\sum_{m=0}^{p}\sum_{n=0}^{q}\sum_{k=0}^{r}\left(\left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k}$$

and

$$A\sigma_{pqr}^{X} = \frac{1}{pqr}\sum_{m=0}^{p}\sum_{n=0}^{q}\sum_{k=0}^{r} a_{mnk}^{pqr}\left(\left(\left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k},0\right).$$

We say that $\left(\frac{X_{mnk}}{Y_{mnk}}\right)$ is statistically lacunary equivalent summable $(C,1,1,1)$ to 0, if the sequence $\sigma = (\sigma_{mnk}^{X})$ is statistically convergent to 0, that is, $st_3 - lim_{pqr}\sigma_{pqr}^{X} = 0$. It is denoted by $C_{111}(st_3)$.

Let $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots + q_{rs}$, $\overline{Q}_s = \overline{q}_{11} + \cdots + \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \cdots + \overline{\overline{q}}_{rs}$.

3. 1. Definition

A triple $(X, P, *)$ be a $PP-$ space. Then a triple sequence $X = (X_{mnk})$ is said to statistically convergent to $\overline{0}$ with respect to the probabilistic $p-$ metric $P-$ provided that for every $\varepsilon > 0$ and $\gamma \in (0,1)$

$$\delta\left(\left\{m,n,k \in \mathbb{N}: P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon)\right] \leq 1-\gamma\right\}\right) = 0$$

or equivalently

$$\begin{aligned} lim_{k\ell v}\frac{1}{k\ell v} m \leq k, n \leq \ell, k \\ \leq v: P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon)\right] \\ \leq 1-\gamma = 0 \end{aligned}$$

In this case we write $St_{PP} - lim_X = 0$.

3. 2. Definition

A triple $(X, P, *)$ be a $PP-$ space. The two non-negative sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are said to be almost asymptotically statistical equivalent of multiple 0 in $PP-$ space X if for every $\varepsilon > 0$ and $\gamma \in (0,1)$.

$$\delta\left(\left\{m, n, k \in \mathbb{N}: P - lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon), \overline{0}\right] \leq 1 - \gamma\right\}\right) = 0$$

or equivalently

$$lim_{k\ell v} \frac{1}{k\ell} \left|\left\{m \leq k, n \leq \ell, k \leq v: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \overline{0}}(\varepsilon) \leq 1 - \gamma\right\}\right| = 0.$$

In this case we write $X \overset{\hat{S}(PP)}{\equiv} Y$.

3. 3. Definition

A triple $(X, P, *)$ be a $PP-$ space and $\theta = (m_r n_s k_t)$ be a lacunary sequence. The two non-negative sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are said to be a almost asymptotically lacunary statistical equivalent of multiple $\overline{0}$ in $PP-$ space X if for every $\varepsilon > 0$ and $\gamma \in (0,1)$

$$\delta_\theta\left(\left\{m, n, k \in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \overline{0}}(\varepsilon) \leq 1 - \gamma\right\}\right) = 0 \qquad (3.2)$$

or equivalently

$$lim_{rst} \frac{1}{h_{rst}} \left|\left\{m, n \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \overline{0}}(\varepsilon) \leq 1 - \gamma\right\}\right| = 0.$$

In this case we write $X \overset{\widehat{S_\theta}(PP)}{\equiv} Y$.

3. 4. Lemma

A triple $(X, P, *)$ be a $PP-$ space. Then for every $\varepsilon > 0$ and $\gamma \in (0,1)$, the following statements are equivalent:

(1) $$lim_{rst} \frac{1}{h_{rst}} \left|\left\{m, n, k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \overline{0}}(\varepsilon) \leq 1 - \gamma\right\}\right| = 0,$$

(2) $$\delta_\theta\left(\left\{m, n, k \in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \overline{0}}(\varepsilon) \leq 1 - \gamma\right\}\right) = 0,$$

(3) $\delta_\theta\left(\left\{m,n,k \in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) \leq 1-\gamma\right\}\right) = 1,$

(4) $lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n,k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) \leq 1-\gamma\right\}\right| = 1.$

4 MAIN RESULTS

4. 1. Theorem

Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. If two triple sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are almost asympototically lacunary statistical equivalent of multiple $\bar{0}$ with respect to the probabilistic $p-$ metric P, then 0 is unique sequence

Proof: Assume that $X \overset{\widehat{S_\theta^{\bar{0}}}(PP)}{\equiv} Y$. For a given $\lambda > 0$ choose $\gamma \in (0,1)$ such that $(1-\gamma) > 1-\lambda$. Then, for any $\varepsilon > 0$, define the following set:

$$K = \left\{m,n,k \in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) \leq 1-\gamma\right\}$$

Then, clearly

$$lim_{rst}\frac{K \cap \ \bar{0}}{h_{rst}} = 1,$$

so K is non-empty set,since $x \overset{\widehat{S_\theta^{\bar{0}}}(PP)}{\equiv} y, \delta_\theta(K) = 0$ for all $\varepsilon > 0$, which implies $\delta_\theta(\mathbb{N}-K) = 1$. If $m,n,k \in \mathbb{N}-K$, then we have

$$P_{\bar{0}}(\varepsilon) = P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > (1-\gamma) \geq 1-\lambda$$

since λ is arbitrary, we get $P_{\bar{0}}(\varepsilon) = 1$.
This completes the proof.

4. 2. Theorem

Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s k_t), \widehat{S_\theta}(PP) \subset \hat{S}(PP)$ if $limsup_{rst} q_{rst} < \infty$.

Proof: If $limsup_{rst} q_{rst} < \infty$. then there exists a $B > 0$ such that $q_{rst} < B$ for all $r,s,t \geq 1$. Let $X \overset{\widehat{S_\theta}(PP)}{\equiv} Y$ and $\varepsilon > 0$. Now we have to prove $\hat{S}(PP)$. Set

$$K_{rst} = \left|\left\{m,n,k \in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right|.$$

Then by definition, for given $\varepsilon > 0$, there exists $r_0 s_0 t_0 \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\varepsilon}{2B} \text{ for all } r > r_0, s > s_0 \text{ and } t > t_0.$$

Let $M = max\{K_{rst}: 1 \le r \le r_0, 1 \le s \le s_0, 1 \le t \le t_0\}$ and let uvw be any positive integer with $m_{r-1} < u \le m_r, n_{s-1} < v \le n_s$ and $k_{t-1} < w \le k_t$.

Then

$$\begin{aligned}
\frac{1}{uvw}&\left|\left\{m \le u, n \le v, k \le w: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \\
&\le \frac{1}{m_{r-1}n_{s-1}k_{t-1}}\left|\left\{m \le m_r, n \le n_s, k \le k_t: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \\
&= \frac{1}{m_{r-1}n_{s-1}k_{t-1}}\{K_{111} + \cdots + K_{rst}\} \le \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2B} q_{rst} \\
&\le \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2}.
\end{aligned}$$

This completes the proof.

4. 3. Theorem

Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s k_t), \hat{S}(PP) \subset \widehat{S_\theta}(PP)$ if $liminf_{rst} q_{rst} > 1$.

Proof: If $liminf_{rst} q_{rst} > 1$, then there exists a $\beta > 0$ such that $q_{rst} > 1+\beta$ for sufficiently large rst, which implies

$$\frac{h_{rst}}{K_{rst}} \ge \frac{\beta}{1+\beta}.$$

Let $X \overset{\widehat{S^{\bar{0}}}(pp)}{\equiv} Y$, then for every $\varepsilon > 0$ and for sufficiently large r, s, t we have

$$\begin{aligned}
&\frac{1}{m_r n_s k_t}\left|\left\{m \le m_r, n \le n_s, k \le k_t: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \ge \\
&\frac{1}{m_r n_s k_t}\left|\left\{m,n,k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \ge \\
&\frac{\beta}{1+\beta}\frac{1}{h_{rst}}\left|\left\{m,n,k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}}{}_{-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right|.
\end{aligned}$$

Therefore $X \overset{\widehat{S^{0}\theta}(pp)}{\equiv} Y$.

This completes the proof.

4. 4. Corollary

Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP -$ space. For any lacunary sequence $\theta = (m_r n_s)$, with $1 < liminf_{rs} q_{rs} \leq limsup_{rst} q_{rst} < \infty$, then $\hat{S}(PP) = \widehat{S_\theta}(PP)$.

Proof: The result clearly follows from Theorem 4.2 and Theorem 4.3.

5. CONCLUSIONS

We introduced the concept of almost lacunary statistical Cesa'ro of Γ^3 over probabilistic p- metric spaces defined by Musielak Orlicz function.The authors feel that this concept in a PP-space would provide a more general framework for the subject, since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis,

ACKNOWLEDGEMENT

The first author and third author wish to thank the Department of Science and Technology, Government of India for the financial sanction towards this work under FIST program SR/FST/MSI-107/2015.

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