Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 96 (2018) 59-82 EISSN 2392-2192

Maxwell-Lorentz Matrix

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ABSTRACT

Lorentz invariance of Maxwell electromagnetic equations is demonstrated in two complementary ways: first, we give a pedestrian review with three-vector equations, and we then express Maxwell equations in a four-vector matrix form (the Maxwell-Lorentz matrix) which demonstrates the intimate connection of Maxwell equations with the Lorentz group. Each Maxwell-Lorentz matrix component is the product of three matrices: a derivative matrix, a 4x4 Lorentz group generator matrix, and an electromagnetic field matrix. We obtain rotary Lorentz transformations of the electromagnetic field matrix from Lorentz equation matrices. We then transform the derivative and electromagnetic matrices and obtain an explicit matrix demonstration of Lorentz invariance of Maxwell equations. To obtain this result, we express all transformation matrices in exponential form to facilitate the application of simple Lorentz group algebra. The pedestrian approach illustrates what the Lorentz group matrix approach actually accomplishes and helps one to gain some appreciation of group theory methods.

***Keywords*:** Lorentz Group, Maxwell equations, Lorentz transformations

(Received 05 March 2018; Accepted 22 March 2018; Date of Publication 23 March 2018)

1. INTRODUCTION: MAXWELL EQUATIONS IN COMPLEX FORM

The relativistic invariance of Maxwell equations continues to be a topic of investigation and clarification. Jefimenko [1] considers scalar expressions of Maxwell equations in their traditional 3-vector form:

$$\nabla \cdot \mathrm{D} = \rho, \qquad \nabla \cdot \mathrm{B} = 0, \qquad \nabla \times \mathrm{H} = \mathrm{J} + \frac{\partial B}{\partial t}, \qquad \nabla \times \mathrm{E} = -\frac{\partial B}{\partial t}, \tag{1}$$

with constitutive equations:

$$\mathrm{D} = \varepsilon_0 \mathrm{E}, \qquad \mathrm{B} = \mu_0 \mathrm{H}, \tag{2}$$

to show that these equations do not transform independently, but this difficulty can be avoided by combining (real) electrical and magnetic fields into the complex Faraday form [2-11]:

$$\mathrm{F} = \mathrm{E} + ic\mathrm{B} \quad , \tag{3}$$

to obtain:

$$\nabla \cdot F = \frac{\rho}{\varepsilon_0}, \qquad -i\nabla \times \mathrm{F} - \frac{\partial \mathrm{F}}{\partial ct} = \frac{j}{c\varepsilon_0}, \tag{4}$$

and we further simplify these equations by defining:

$$\psi = c\varepsilon_0 \mathrm{F}, \tag{5}$$

to obtain Maxwell equations in a succinct three-vector form as:

$$\nabla \cdot \psi = \rho c, \qquad \mathrm{M} = \mathrm{j}, \tag{6a}$$

where we define:

$$\mathrm{M} \equiv -i\nabla \times \psi - \frac{\partial \psi}{\partial ct}. \tag{6b}$$

We refer to these as Maxwell ψ-form equations, and recalling that $j_t = \rho c$ is the time component of the current density four-vector, we can write the four Maxwell component expressions of an x, y, z Cartesian system in column (ket) matrices as:

$$\left|M\right\rangle = \left|j\right\rangle, \qquad \text{i.e.} \quad \begin{pmatrix} M_t \\ M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} j_t \\ j_x \\ j_y \\ j_z \end{pmatrix}, \tag{7}$$

where we refer to $\left|M\right\rangle$ as the Maxwell matrix, which contains four components:

$$M_t = \nabla \cdot \psi = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}, \qquad M_x = \hat{x} \cdot \mathrm{M} = -\frac{\partial \psi_x}{\partial ct} - i\frac{\partial \psi_z}{\partial y} + i\frac{\partial \psi_y}{\partial z},$$
$$M_y = \hat{y} \cdot \mathrm{M} = -\frac{\partial \psi_y}{\partial ct} + i\frac{\partial \psi_z}{\partial x} - i\frac{\partial \psi_x}{\partial z}, \qquad M_z = \hat{z} \cdot \mathrm{M} = -\frac{\partial \psi_z}{\partial ct} - i\frac{\partial \psi_y}{\partial x} + i\frac{\partial \psi_x}{\partial y}. \tag{8}$$

Since there is no ψ_t in these equations, we can consider them to be three-vector expressions, but $\left|j\right\rangle$ is a well-known four vector, so we can see that Maxwell matrix $\left|M\right\rangle$ is also a four-vector.

A complete collection of equations adequate for relativistic transformation of Maxwell equations is given by Lorrain et al [12, 13]. We adopt their convention that the primed system is moving with velocity **v** with respect to the unprimed system. In several books that treat relativity with group theory [14-16], the equations show that their primed coordinate system moves with velocity –**v** with respect to the unprimed system (in spite of any illustrations to the contrary). For simplicity, we consider the respective axes of the primed and unprimed systems to be aligned, and we consider the velocity **v** to be parallel to the *x, y*, or *z*-axis (i.e., an *x, y*, or *z*-boost).

We write Maxwell equations in the primed system by simply putting a prime on all the symbols (except *i* and *c*), so we can write:

$$\left|M'\right\rangle = \left|j'\right\rangle, \tag{9}$$

and inquire whether this implies eq. (7).

A major advantage of four-vectors is that their Lorentz transformation can be expressed with simple 4x4 matrices, and Lorentz transformation of real four-vector matrices like the current density $\left|j\right\rangle$ are done with the Lorentz boost matrices $b_k(\beta)$ where k = 1, 2, 3 refers to a boost in the *x, y, z*-direction, so:

$$\left| j' \right\rangle = b_k(\beta) \left| j \right\rangle, \qquad \text{i.e.} \qquad \begin{pmatrix} j'_t \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = b_k(\beta) \begin{pmatrix} j_t \\ j_x \\ j_y \\ j_z \end{pmatrix}, \tag{10}$$

where:

$$b_1(\beta) = b_x(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b_2(\beta) = b_y(\beta) = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b_3(\beta) = b_z(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{11}$$

the usual relativistic parameters in these matrices are $\beta = v/c$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$.

We have identified Maxwell matrix $\left| M \right\rangle$ as a four-vector, and we now realize that it probably transforms in the same way as $\left| j \right\rangle$, so we conjecture that:

$$\left| M' \right\rangle = b_k(\beta) \left| M \right\rangle. \tag{12a}$$

If this is true, then $\left| M' \right\rangle = \left| j' \right\rangle$, implies that:

$$b_k(\beta) \left| M \right\rangle = b_k(\beta) \left| j \right\rangle, \tag{12b}$$

and we can multiply both sides by $b_k(-\beta)$ to obtain eq.(7), which would show the Lorentz invariance of Maxwell equations.

In this paper, we explicitly demonstrate that $\left| M' \right\rangle = b_k(\beta) \left| M \right\rangle$ in two complementary ways: we first provide an opportunity to gain an intuitive understanding of Lorentz invariance by directly substituting Lorentz transformation equations into Maxwell equations in ψ-form. We then express Maxwell equations in a matrix form which shows the origin of the conventional electromagnetic tensors. Following these preliminaries, we demonstrate the intimate connection of Maxwell equations with the Lorentz group.

We define the Maxwell-Lorentz matrix, transform the electromagnetic field and derivative matrices with rotary and inverse-boost Lorentz transformations, and explicitly demonstrate invariance of the Maxwell-Lorentz matrix expression by using simple group algebra. We conclude by explicitly connecting our development with the Moses [17] matrix expansion, and this procedure reveals an important symmetry of the group matrices.

2. LORENTZ TRANSFORMATION OF THE ELECTROMAGNETIC FIELD WITH ROTATION MATRICES

Strange [18] gives a nice discussion of the Lorentz transformation, and we combine his electrical and magnetic field transformation equations for an x-boost to obtain the simple relation:

$$\begin{pmatrix} E'_x + icB'_x \\ E'_y + icB'_y \\ E'_z + icB'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & i\beta\gamma \\ 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_x + icB_x \\ E_y + icB_y \\ E_z + icB_z \end{pmatrix}. \tag{13}$$

An interesting thing about Lorentz transformation of the electromagnetic field is that the component in the boost direction is invariant, unlike the Lorentz boost transformation where the transverse components are invariant. Armour [10] gives many references which have combined electrical and magnetic fields into the Riemann-Silberstein vector, and keeping Armour´s discussion in mind, we recall our ψ-vector and define a four-component $|\psi\rangle$ electromagnetic matrix as:

$$|\psi\rangle = \begin{pmatrix} \psi_t \\ \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} = c\varepsilon_0 \begin{pmatrix} \Gamma \\ E_x + icB_x \\ E_y + icB_y \\ E_z + icB_z \end{pmatrix} \quad , \tag{14a}$$

where: Γ is an invariant gauge-fixing condition parameter:

$$\Gamma = \frac{\partial}{\partial ct}\left(\frac{\Phi}{c}\right) + \nabla \cdot \mathrm{A}. \tag{14b}$$

The parameter Γ is zero in the Lorenz gauge (not a typo, see the discussion of Lorenz and Lorentz by Jackson [19, 20]), and (Φ/c, **A**) is the four-vector electromagnetic potential. We rewrite our x-boost electromagnetic transformation of eq.(13) as:

$$|\psi'\rangle = h_x(\beta)|\psi\rangle, \qquad \text{i.e.} \qquad \begin{pmatrix} \psi'_t \\ \psi'_x \\ \psi'_y \\ \psi'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \psi_t \\ \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}, \tag{15a}$$

and more generally we write the electromagnetic transformation for a boost parallel to the k-axes (where k = 1, 2, 3 = x, y, z) as:

$$|\psi'\rangle = h_k(\beta)|\psi\rangle, \tag{15b}$$

where the $h_k(\beta)$ are 4 x 4 Hermitian matrices (equal to their transpose conjugate) which are definitely not Lorentz boosts. One may question wheter these transformations qualify as Lorentz transformations, so we now show that the electromagnetic transformation matrices $h_k(\beta)$ can be obtained from Lorentz group rotation matrices.

We recall that the boost matrices are parameterized in group theory [14-16] with an angle, and we follow this approach by defining ξ so that sinhξ = $\beta\gamma$ and coshξ = γ, with the result that tanhξ = β; i.e., the velocity determines the angle ξ so that the boost matrix:

$$b_x(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Our x-boost electromagnetic transformation (not a boost matrix!) is:

$$h_x(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}, \tag{17a}$$

and using the group theory parameterization acquired from the boost matrices, we obtain:

$$h_x(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\xi & i\sinh\xi \\ 0 & 0 & -i\sinh\xi & \cosh\xi \end{pmatrix}, \tag{17b}$$

but we now recall that the Lorentz group rotation matrix about the x-direction is:

$$R_x(\alpha)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\alpha&\sin\alpha\\0&0&-\sin\alpha&\cos\alpha\end{pmatrix}, \tag{18a}$$

and substituting $\alpha = i\xi$, we find that:

$$R_x(i\xi)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos(i\xi)&\sin(i\xi)\\0&0&-\sin(i\xi)&\cos(i\xi)\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cosh\xi&i\sinh\xi\\0&0&-i\sinh\xi&\cosh\xi\end{pmatrix}=h_x(\beta), \tag{18b}$$

and more generally we find that:

$$h_k(\beta)=R_k(i\xi). \tag{18c}$$

we have found that the electromagnetic transformation for a boost in the *k*-direction can be obtained from the corresponding Lorentz rotation matrix about the *k*-axis, and this procedure establishes the electromagnetic transformation as a legitimate Lorentz transformation which we refer to as a rotary Lorentz transformation.

3. LORENTZ TRANSFORMATION OF THE DERIVATIVE MATRICES WITH INVERSE BOOST MATRICES

We define the ket (column) matrix $|\partial\rangle$ as:

$$|\partial\rangle=\begin{pmatrix}\frac{\partial}{\partial ct}\\ \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}, \tag{19}$$

and find [12. 13] that $|\partial\rangle$ transforms for a *k*-boost with an inverse *k*-boost matrix as:

$$|\partial'\rangle=b_k(-\beta)|\partial\rangle, \tag{20}$$

so for a *z*-boost, the derivative matrix transforms as:

$$\left|\partial'\right\rangle = b_z(-\beta)\left|\partial\right\rangle, \qquad \text{i.e.} \qquad \begin{pmatrix} \frac{\partial}{\partial ct'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial ct} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}. \tag{21}$$

This section completes the discussion of transformation equations necessary to prove Lorentz invariance of Maxwell equations by direct substitution into the four component equations, and this procedure is suggested as a tedious but instructive approach, and it is done in the next section for a *z*-boost Lorentz transformation, but can also be done for an *x*- or *y*-boost. The labor required for this approach should be enough to convince one that there must be a better way, so we will do the same study with matrices and Lorentz group theory.

4. LORENTZ INVARIANCE OF MAXWELL EQUATIONS (FOR A BOOST IN THE Z-DIRECTION) BY DIRECT SUBSTITUTION

Assuming that Maxwell equations are true in the primed system, we now substitute Lorentz transformation scalar equations (contained within the field and derivative transformation matrices) into the primed component Maxwell equations to show that these equations also hold in the unprimed system. We arbitrarily choose to do this for a boost in the *z*-direction.

In the primed system, we first consider $M'_t = \nabla'\cdot\psi' = \left(\frac{\partial}{\partial x'}\right)\psi'_x + \left(\frac{\partial}{\partial y'}\right)\psi'_y + \left(\frac{\partial}{\partial z'}\right)\psi'_z$, and substitute the Lorentz transformation equations for a *z*-boost to obtain:

$$\begin{aligned} M'_t &= \left(\frac{\partial}{\partial x}\right)\gamma\left(\psi_x + i\beta\gamma\psi_y\right) + \left(\frac{\partial}{\partial y}\right)\gamma\left(\psi_y - i\beta\psi_x\right) + \gamma\left(\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial ct}\right)\psi_z \\ &= \gamma\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y} + \frac{\partial\psi_z}{\partial z}\right) - \beta\gamma\left(-i\frac{\partial\psi_y}{\partial x} + i\frac{\partial\psi_x}{\partial y} - \frac{\partial\psi_z}{\partial ct}\right) = \gamma M_t - \beta\gamma M_z. \end{aligned}$$

For a *z*-boost, we find that M'_t is a linear combination of M_t and M_z, the boost direction component of Maxwell equations.

We then consider the boost direction component:

$$M'_z = -i\frac{\partial \psi'_y}{\partial x'} + i\frac{\partial \psi'_x}{\partial y'} - \frac{\partial \psi'_z}{\partial ct'} = -i\left(\frac{\partial}{\partial x'}\right)\psi'_y + i\left(\frac{\partial}{\partial y'}\right)\psi'_x - \left(\frac{\partial}{\partial ct'}\right)\psi'_z,$$

and we substitute the Lorentz transformation equations to obtain:

$$M'_z = -i\left(\frac{\partial}{\partial x}\right)\gamma\left(\psi_y - i\beta\psi_x\right) + i\left(\frac{\partial}{\partial y}\right)\gamma\left(\psi_x + i\beta\psi_y\right) - \gamma\left(\frac{\partial}{\partial ct} + \beta\frac{\partial}{\partial z}\right)\psi_z,$$
$$= -\beta\gamma\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}\right) + \gamma\left(-i\frac{\partial \psi_y}{\partial x} + i\frac{\partial \psi_x}{\partial y} - \frac{\partial \psi_z}{\partial ct}\right) = -\beta\gamma M_t + \gamma M_z.$$

We have transformed the left side of two equations, and we find that each of these is transformed into linear combination of M_t and M_z, so we can treat M_t and M_z as unknowns and simultaneously solve these two equations $\gamma M_t - \beta\gamma M_z = \rho' c$, $-\beta\gamma M_t + \gamma M_z = j'_z$ to obtain $M_t = \frac{\rho' c + \beta j'_z}{\gamma\left(1-\beta^2\right)} = \rho c$, $M_z = \frac{j'_z + \beta\rho' c}{\gamma\left(1-\beta^2\right)} = j_z$, where we transform the primed expressions on the right side as the last step. We have proven the Lorentz invariance of the *t*- and *z*-component equations under a *z*-boost.

The components of **M** transverse to the *z*-boost are uncoupled, so these transform independently to an invariant form. We substitute the Lorentz transformation equations into $M'_x = -i\left(\frac{\partial}{\partial y'}\right)\psi'_z + i\left(\frac{\partial}{\partial z'}\right)\psi'_y - \left(\frac{\partial}{\partial ct'}\right)\psi'_x$ to obtain:

$$M'_x = -i\left(\frac{\partial}{\partial y}\right)\psi_z + i\gamma\left(\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial ct}\right)\gamma\left(\psi_y - i\beta\psi_x\right) - \gamma\left(\frac{\partial}{\partial ct} + \beta\frac{\partial}{\partial z}\right)\gamma\left(\psi_x + i\beta\psi_y\right)$$
$$= -i\frac{\partial \psi_z}{\partial y} + i\frac{\partial \psi_y}{\partial z} - \frac{\partial}{\partial ct}\psi_x = M_x,$$

and $j'_x = j_x$, so $M'_x = j'_x$ is Lorentz transformed under a *z*-boost directly into $M_x = j_x$. In the same way, $M'_y = j'_y$ transforms directly into $M_y = j_y$ under a *z*-boost.

We find that our four scalar Maxwell equations are invariant under Lorentz transformation. However, the key point is that transverse component equations transform independently while the time-component equation couples to the boost-direction equation under Lorentz transformation, and these two coupled equations can be solved simultaneously to prove Lorentz invariance (of each equation). Alternatively, we can place our four transformed equations in the Maxwell matrix to obtain:

$$\begin{pmatrix} M'_t \\ M'_x \\ M'_y \\ M'_z \end{pmatrix} = \begin{pmatrix} \gamma M_t - \beta\gamma M_z \\ M_x \\ M_y \\ -\beta\gamma M_t + \gamma M_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} M_t \\ M_x \\ M_y \\ M_z \end{pmatrix} = b_z(\beta) \begin{pmatrix} M_t \\ M_x \\ M_y \\ M_z \end{pmatrix}, \qquad (22)$$

but this is eq. (12a) for a *z*-boost, and as discussed there, this result establishes Lorentz invariance of Maxwell equations under a *z*-boost. We could establish invariance under *x*- and *y*-boosts in the same way, but we seek to obtain more insight with less labor, so we now express Maxwell equations in matrix form so that the required Lorentz transformations (boost, inverse boost, and rotary) can all be done with our 4 x 4 matrices.

5. MAXWELL EQUATIONS IN MATRIX FORM

We now seek to express Maxwell equations in matrix form, so recalling that $\psi_t = 0$ in the Lorenz gauge, we add $\frac{\partial \psi_t}{\partial ct}$ to the left side of the *t*-component scalar equation and substract $\frac{\partial \psi_t}{\partial x_\mu}$ from the μ = 1, 2, 3 component scalar equations to obtain Maxwell scalar equations as:

$$\frac{\partial \psi_t}{\partial ct} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z} = j_t, \qquad -\frac{\partial \psi_x}{\partial ct} - \frac{\partial \psi_t}{\partial x} - i\frac{\partial \psi_z}{\partial y} + i\frac{\partial \psi_y}{\partial z} = j_x,$$
$$-\frac{\partial \psi_y}{\partial ct} + i\frac{\partial \psi_z}{\partial x} - \frac{\partial \psi_t}{\partial y} - i\frac{\partial \psi_x}{\partial z} = j_y, \qquad -\frac{\partial \psi_z}{\partial ct} - i\frac{\partial \psi_y}{\partial x} + i\frac{\partial \psi_x}{\partial y} - \frac{\partial \psi_t}{\partial z} = j_z. \qquad (23)$$

Since one may question our procedure of adding a term to the *t*-component equation while subtracting from the other components, we can alternatively describe the procedure as adding a term:

$$T_\mu = g_{\mu\mu} \frac{\partial \psi_t}{\partial x_\mu}, \qquad (24a)$$

to each μ-component equation (with no summation over μ implied on the right). The $g_{\mu\mu}$ are the nonzero (diagonal) elements of the metric tensor:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{so} \quad g_{\mu\kappa} = \delta_{\mu\kappa}(2\delta_{\mu,0} - 1), \text{ and } \quad T_\mu = (2\delta_{\mu,0} - 1)\frac{\partial \psi_t}{\partial x_\mu}. \tag{24b}$$

We now seek to express the left hand side of each scalar equation in matrix form, so we transpose the column (ket) derivative matrix to obtain the row (bra) derivative matrix:

$$\langle \partial | = | \partial \rangle^T = \begin{pmatrix} \frac{\partial}{\partial ct} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}, \tag{25}$$

and we now write the components of Maxwell matrix as:

$$M_\mu = \langle \partial || \psi^\mu \rangle, \tag{26}$$

where we compare eq. (41) with eqs. (23) to find that the column matrices $|\psi_\mu\rangle$ are:

$$|\psi^0\rangle = |\psi^t\rangle = \begin{pmatrix} \psi_t \\ \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}, \ |\psi^1\rangle = |\psi^x\rangle = \begin{pmatrix} -\psi_x \\ -\psi_t \\ -i\psi_z \\ i\psi_y \end{pmatrix}, |\psi^2\rangle = |\psi^y\rangle = \begin{pmatrix} -\psi_y \\ i\psi_z \\ -\psi_t \\ -i\psi_x \end{pmatrix}, |\psi^3\rangle = |\psi^z\rangle = \begin{pmatrix} -\psi_z \\ -i\psi_y \\ i\psi_x \\ -\psi_t \end{pmatrix}. \tag{27}$$

We combine these scalar equations in the Maxwell column matrix as:

$$|M\rangle = \begin{pmatrix} \langle \partial | \psi^0 \rangle | \\ \langle \partial | \psi^1 \rangle | \\ \langle \partial | \psi^2 \rangle | \\ \langle \partial | \psi^3 \rangle | \end{pmatrix}, \tag{28}$$

and we will proceed from this equation into Lorentz group theory after briefly digressing into the traditional electromagnetic tensors.

6. THE ELECTROMAGNETIC TENSORS

Transposing eq.(7) and eq.(28), we can rewrite Maxwell equations in row (bra) matrix form as:

$$\langle M| = \langle j| \tag{29}$$

where:

$$\langle M| = \left(\langle \partial|\psi^0\rangle| \quad \langle \partial||\psi^1\rangle \quad \langle \partial|\psi^2\rangle| \quad \langle \partial|\psi^3\rangle| \right) = \langle \partial|\boldsymbol{\Psi}, \tag{30}$$

$$\boldsymbol{\Psi} = \left(|\psi^0\rangle \quad |\psi^1\rangle \quad |\psi^2\rangle \quad |\psi^3\rangle \right) = \begin{pmatrix} \psi_t & -\psi_x & -\psi_y & -\psi_z \\ \psi_x & -\psi_t & i\psi_z & -i\psi_y \\ \psi_y & -i\psi_z & -\psi_t & i\psi_x \\ \psi_z & i\psi_z & -i\psi_x & -\psi_t \end{pmatrix}, \tag{31}$$

and:

$$\langle j| = \left(j_0 \quad j_1 \quad j_2 \quad j_3 \right). \tag{32}$$

Expressing **ψ** as the sum of real and imaginary parts, we can write:

$$\boldsymbol{\Psi} = \mathrm{Re}\boldsymbol{\Psi} + i\,\mathrm{Im}\boldsymbol{\Psi} = c\varepsilon_0(\mathrm{T} + i\mathrm{T}_d), \tag{33a}$$

where **T** is the electromagnetic field tensor:

$$\mathrm{T} = \frac{1}{c\varepsilon_0}\mathrm{Re}\boldsymbol{\Psi} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}, \tag{33b}$$

and $\mathbf{T}_d$ is the dual electromagnetic field tensor:

$$\mathrm{T}_d = \frac{1}{c\varepsilon_0}\mathrm{Im}\boldsymbol{\Psi} = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{pmatrix}, \tag{33c}$$

so the tensor equation:

$$\langle \partial|\mathrm{T} = \frac{1}{c\varepsilon_0}\langle j|, \tag{34a}$$

gives the four scalar equations which correspond to $\nabla \cdot \mathrm{E} = \frac{\rho c}{c\varepsilon_0}$, and $\nabla \times c\mathrm{B} - \frac{\partial \mathrm{E}}{\partial ct} = \frac{j}{c\varepsilon_0}$, while the tensor equation:

$$\langle \partial | \mathrm{T}_d = 0, \tag{34b}$$

which gives the four scalar equations corresponding to $\nabla \cdot c\mathrm{B} = 0$, and $\nabla \times \mathrm{E} = -\frac{\partial c\mathrm{B}}{\partial ct}$.

Having expressed Maxwell equations with the traditional electromagnetic tensors, we now comment that these forms seem somewhat unfortunate since they obscure the Lorentz group structure.

7. LORENTZ GROUP GENERATOR MATRICES

We now introduce Lorentz group theory and express Maxwell equations with matrix generators of the Lorentz group. We start by defining four (4 x 4) matrices G^μ with the equation:

$$G^\mu |\psi\rangle = |\psi^\mu\rangle, \tag{35}$$

so that we can rewrite eq. (26) as:

$$M_\mu = \langle \partial | G^\mu | \psi \rangle, \tag{36}$$

where μ = 0, 1, 2, 3 identifies the *t*, *x*, *y*, *z* scalar Maxwell equations. Comparing eq. (35) with eq. (26) allows one to determine the G^μ matrices to be:

$$G^0 = G^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad G^1 = G^x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{37a}$$

$$G^2 = G^y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad G^3 = G^z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{37b}$$

We now seek to determine the identity and significance of the G^μ matrices. Ryder [16] gives a section on SL(2,C) and the Lorentz group, and with Ryder´s matrices as a guide, we find that our G^μ matrices are:

$$G^\mu = J^\mu + iK^\mu, \tag{38}$$

where the J^μ matrices, defined as:

$$J^\mu = (-i)\left[\frac{\partial}{\partial\alpha}R_\mu(\alpha)\right]_{\alpha=0}, \tag{39a}$$

are generators of the Lorentz group rotation matrices, which means that the rotation matrices can be expressed exponentially as:

$$R_\mu(\alpha) = \exp(iJ^\mu\alpha). \tag{39b}$$

Meanwhile, the K^μ matrices, defined as:

$$iK^\mu = \left(\frac{\partial}{\partial\xi}b_z(\beta)\right)_{\xi=0}, \tag{40a}$$

are generators of the Lorentz boost matrices in the ξ parameterization, which means that the boost matrices can be expressed exponentially as:

$$b_\mu(\beta) = \exp(iK^\mu\xi), \tag{40b}$$

and an inverse boost is:

$$b_\mu(-\beta) = \exp(-iK^\mu\xi). \tag{40c}$$

Our J^μ matrices are the same as Ryder´s but our iK^μ matrices differ from his by a factor of (-1) because his boost matrix is for a primed system moving with velocity of –**v** relative to the unprimed system (as mentioned above) rather than **v**, which we use while following the traditional standard for Lorentz transformations.

Why does the generator of a rotation group matrix appear in Lorentz group theory? The answer is that the three generalized 4 x 4 rotation matrices (for rotations about the *x, y,* and *z*-axes are legitimate Lorentz transformations (i.e., members of the Lorentz group), and the inclusion of the rotation matrices with the boost matrices is required to complete the Lorentz group since the boost matrices alone do not constitute a group [16]. Another consideration is that the Lorentz electromagnetic transformations are obtained from rotation matrices, as noted in eq. (18c) $h_\kappa(\beta) = R_\kappa(i\xi)$, and the group theory result of eq.(39b), $R_\kappa(\alpha) = \exp(iJ^\kappa\alpha)$, gives the electromagnetic transformation matrices in exponential form as:

$$h_\kappa(\beta) = R_\kappa(i\xi) = \exp(-J^\kappa\xi). \tag{41}$$

We have found that each scalar Maxwell equations is determined by the corresponding Lorentz group generator matrix $G^\mu = J^\mu + iK^\mu$, where μ = 0, 1, 2, 3 = *t, x, y, z* is the number of the scalar equation. The generator matrices G^μ allow complicated expressions to be greatly simplified because they obey the same (simple) SU(2) algebra as the Pauli spin matrices. In Appendix A, we list the G^{μ} multiplication rules and the exponential expressions of the Lorentz inverse boost and rotary matrices. In the other Appendices, we then develop a sequence of short problems to obtain commutation relations with which we prove Lorentz invariance of Maxwell equations in the next section.

8. LORENTZ INVARIANCE OF MAXWELL EQUATIONS USING GROUP THEORY

Recalling eq.(36), $M_{\mu} = \langle \partial | G^{\mu} | \psi \rangle$, we define the Maxwell-Lorentz matrix as:

$$|M\rangle = \begin{pmatrix} \langle \partial | G_0 | \psi \rangle \\ \langle \partial | G_1 | \psi \rangle \\ \langle \partial | G_2 | \psi \rangle \\ \langle \partial | G_3 | \psi \rangle \end{pmatrix}. \tag{42}$$

We now proceed to demonstrate Lorentz invariance of Maxwell equations for a *k*-boost by transforming the primed Maxwell-Lorentz matrix:

$$|M'\rangle = \begin{pmatrix} \langle \partial' | G_0 | \psi' \rangle \\ \langle \partial' | G_1 | \psi' \rangle \\ \langle \partial' | G_2 | \psi' \rangle \\ \langle \partial' | G_3 | \psi' \rangle \end{pmatrix}, \tag{43}$$

into the unprimed system. The field transformation is given by eq. (15c) $|\psi'\rangle = h_k(\beta)|\psi\rangle$, and we transpose eq. (20) $|\partial'\rangle = b_k(-\beta)|\partial\rangle$, to obtain:

$$\langle \partial' | = \langle \partial | \left(b_{\kappa}(-\beta) \right)^{T} = \langle \partial | b_{\kappa}(-\beta). \tag{44}$$

Substituting these Lorentz transformation equations into eq. (43), we find that the primed Maxwell-Lorentz matrix is:

$$\left|M'\right\rangle = \begin{pmatrix} \langle\partial'|G^0|\psi'\rangle \\ \langle\partial'|G^1|\psi'\rangle \\ \langle\partial'|G^2|\psi'\rangle \\ \langle\partial'|G^3|\psi'\rangle \end{pmatrix} = \begin{pmatrix} \langle\partial|b_\kappa(-\beta)G^0h_\kappa(\beta)|\psi\rangle \\ \langle\partial|b_\kappa(-\beta)G^1h_\kappa(\beta)|\psi\rangle \\ \langle\partial|b_\kappa(-\beta)G^2h_\kappa(\beta)|\psi\rangle \\ \langle\partial|b_\kappa(-\beta)G^3h_\kappa(\beta)|\psi\rangle \end{pmatrix}. \tag{45}$$

To simplify this matrix expression, we recall the exponential expression of the rotary matrices (for k = direction of boost = 1, 2, 3 = x, y, z) given by eq.(41) $h_\kappa(\beta) = \exp(-J^\kappa\xi)$, and we recall the exponential expression of the inverse-boost matrices given by eq. (40c) $b_\kappa(-\beta) = \exp(-iK^\kappa\xi)$.

For convenience, we list these exponential expressions in Appendix A, and we then proceed to obtain the required group theory expressions including:

$$b_\kappa(-\beta)G^0h_\kappa(\beta) = \gamma G^0 - \beta\gamma G^\kappa. \tag{46a}$$

in Appendix C,

$$b_\kappa(-\beta)G^\mu h_\kappa(\beta) = -\beta\gamma G^0 + \gamma G^\kappa \text{ when } \mu = k. \tag{46b}$$

in Appendix D, and:

$$b_\kappa(-\beta)G^\mu h_\kappa(\beta) = G^\mu \quad \text{for } \mu = 1, 2, 3 \quad \text{when } \mu \neq k, \tag{46c}$$

in Appendix F. In matrix form, the utility of these equations becomes clear, as we substitute them into eq.(45) for a boost (with k = 3) and obtain:

$$\begin{pmatrix} \langle\partial'|G^0|\psi'\rangle \\ \langle\partial'|G^1|\psi'\rangle \\ \langle\partial'|G^2|\psi'\rangle \\ \langle\partial'|G^3|\psi'\rangle \end{pmatrix} = \begin{pmatrix} \langle\partial|\left(\gamma G^0 - \beta\gamma G^3\right)|\psi\rangle \\ \langle\partial|G^1|\psi\rangle \\ \langle\partial|G^2|\psi\rangle \\ \langle\partial|\left(-\beta\gamma G^0 + \gamma G^3\right)|\psi\rangle \end{pmatrix} = \begin{pmatrix} \gamma\langle\partial|G^0|\psi\rangle - \beta\gamma\langle\partial|G^3|\psi\rangle \\ \langle\partial|G^1|\psi\rangle \\ \langle\partial|G^2|\psi\rangle \\ -\beta\gamma\langle\partial|G^0|\psi\rangle + \gamma\langle\partial|G^3|\psi\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \langle\partial|G^0|\psi\rangle \\ \langle\partial|G^1|\psi\rangle \\ \langle\partial|G^2|\psi\rangle \\ \langle\partial|G^3|\psi\rangle \end{pmatrix} = b_z(\beta) \begin{pmatrix} \langle\partial|G^0|\psi\rangle \\ \langle\partial|G^1|\psi\rangle \\ \langle\partial|G^2|\psi\rangle \\ \langle\partial|G^3|\psi\rangle \end{pmatrix},$$

and for a k-direction boost, we obtain the more general expression:

$$\begin{pmatrix} \langle \partial' | G^0 | \psi' \rangle \\ \langle \partial' | G^1 | \psi' \rangle \\ \langle \partial' | G^2 | \psi' \rangle \\ \langle \partial' | G^3 | \psi' \rangle \end{pmatrix} = b_\kappa(\beta) \begin{pmatrix} \langle \partial | G^0 | \psi \rangle \\ \langle \partial | G^1 | \psi \rangle \\ \langle \partial | G^2 | \psi \rangle \\ \langle \partial | G^3 | \psi \rangle \end{pmatrix}. \tag{47}$$

We find that the primed Maxwell-Lorentz matrix simplifies to a boost of the unprimed matrix which confirms our application of non-boost Lorentz transformations. Recalling that $| j' \rangle = b_\kappa(\beta) | j \rangle$, we find that the Maxwell-Lorentz expression in the primed system

$\begin{pmatrix} \langle \partial' | G^0 | \psi' \rangle \\ \langle \partial' | G^1 | \psi' \rangle \\ \langle \partial' | G^2 | \psi' \rangle \\ \langle \partial' | G^3 | \psi' \rangle \end{pmatrix} = | j' \rangle$ Lorentz transforms to $b_\kappa(\beta) \begin{pmatrix} \langle \partial | G^0 | \psi \rangle \\ \langle \partial | G^1 | \psi \rangle \\ \langle \partial | G^2 | \psi \rangle \\ \langle \partial | G^3 | \psi \rangle \end{pmatrix} = b_\kappa(\beta) | j \rangle$ and multiplying both

sides of this equation from the left by $b_k(-\beta)$ gives the Maxwell-Lorentz expression in the

unprimed system, $\begin{pmatrix} \langle \partial | G^0 | \psi \rangle \\ \langle \partial | G^1 | \psi \rangle \\ \langle \partial | G^2 | \psi \rangle \\ \langle \partial | G^3 | \psi \rangle \end{pmatrix} = | j \rangle$. We find that this procedure proves Lorentz invariance of

Maxwell equations using group theory matrix expressions obtained from the SU(2) algebra of the G^μ generator matrices and the Lorentz transformation matrices expressed in exponential form.

9. CONNECTION OF THE MAXWELL-LORENTZ EXPRESSION WITH THE MOSES DERIVATIVE EXPANSION

Armour [10] gives extensive references to Moses [17] and other researchers who have used the Riemann-Silberstein vector and the Moses [17] derivative expansion in group theory of Maxwell equations [21]. We now show the connection of these topics with the Maxwell-Lorentz matrix and identify a symmetry of the Hermitian and unitary generator matrices. To facilitate the demonstration of this connection, we represent the Riemann-Silberstein vector as:

$$|C\rangle = \begin{pmatrix} \psi_t \\ -\psi_x \\ -\psi_y \\ -\psi_z \end{pmatrix}, \tag{48}$$

and recognize its relation to our $|\psi\rangle$ matrix as:

$$|\psi\rangle = g|C\rangle, \tag{49}$$

where we recall that g is the metric tensor matrix given by eq.(24b). We then substitute this expression into eq.(36) to obtain:

$$M_\mu = \langle\partial|G^\mu g|C\rangle = \langle\partial|U^\mu|C\rangle = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\partial_\alpha U^\mu_{\alpha\beta}C_\beta \tag{50a}$$

where we define the matrix:

$$U^\mu = G^\mu g. \tag{50b}$$

We recall that the G^μ matrices are Hermitian but observe that the U^μ matrices are unitary. We also find a very interesting symmetry that exists between the *U* and *G* matrices, which is that:

$$U^\mu_{\alpha\beta} = G^\alpha_{\mu\beta}, \tag{51}$$

i.e., the α row of matrix U^μ is the μ row of matrix G^α. The result is that:

$$M_\mu = \sum_{\beta=0}^{3}\sum_{\alpha=0}^{3}\partial_\alpha G^\alpha_{\mu\beta}C_\beta = \sum_{\beta=0}^{3} D_{\mu\beta}C_\beta\ , \tag{52a}$$

where we define:

$$D_{\mu\beta} = \sum_{\alpha=0}^{3}\partial_\alpha G^\alpha_{\mu\beta}, \qquad \text{so that} \qquad D = \sum_{\alpha=0}^{3}\partial_\alpha G^\alpha. \tag{52b}$$

We find that *D* is the linear combination of G^α matrices given by Moses [17] and used recently by Armour [10] and the Maxwell-Lorentz matrix can be written succinctly as:

$$|M\rangle = D|C\rangle. \tag{53}$$

We find that our Maxwell-Lorentz matrix and the Moses expansion are essentially equivalent expressions, and we expect that each of these expressions will have advantages for some applications. In this paper, we give an explicit proof of Lorentz invariance using the Hermitian G^{μ} matrices and its associated algebra. In light of the simple relation $U^{\mu} = G^{\mu} g.$, we now see that the G^{μ} algebra can be extended to the unitary U^{μ} matrices, but that was probably not obvious until now.

10. CONCLUSIONS

The objective of this paper has been to give an introduction to the group theory of Maxwell equations by demonstrating the intimate connection of Maxwell equations with the Lorentz group. We define a Maxwell-Lorentz matrix expression of Maxwell equations using 4 x 4 Lorentz group generator matrices, and we then represent the 4 x 4 Lorentz transformation matrices (boost, inverse-boost, rotation, and electromagnetic) in exponential forms (with another application of the Lorentz group generator matrices). This procedure allows us to apply Lorentz group algebra to products of the inverse boost, G^{μ} generator, and electromagnetic transformation matrices, and the result is an explicit matrix demonstration of Lorentz invariance of Maxwell equations. We show the connection of our approach to the conventional electromagnetic tensors, and we also show the connection with an established expansion and thereby discover an interesting symmetry of the Hermitian and unitary generator matrices. Appendices are designed to serve as applications in the Lorentz group algebra.

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APPENDIX A: G^{μ} MULTIPLICATION RULES AND EXPONENTIAL EXPRESSIONS OF THE LORENTZ TRANSFORMATION MATRICES

In this appendix, we list the G^{μ} multiplication rules and exponential expressions of the Lorentz transformation matrices. These expressions are then used in the following Appendices to obtain expressions showing Lorentz invariance of Maxwell equations expressed with the Maxwell-Lorentz matrix. The G^{μ} matrices obey the multiplication rules:

$$G^{\mu}G^{0} = G^{\mu} \qquad \text{for all } \mu, \tag{A1a}$$

$$G^{\mu}G^{\kappa} = G^{0} = I \quad \text{when} \quad \mu, k = 0, 1, 2, 3 \quad \text{and} \quad \mu = k, \tag{A1b}$$

and:

$$G^{\mu}G^{\kappa} = -G^{\kappa}G^{\mu} = i\sum_{\eta=1,2,3} G^{\eta}\varepsilon_{\mu\kappa\eta} \quad \text{when} \quad \mu, k = 1, 2, 3 \quad \text{and} \quad \mu \neq k. \tag{A1c}$$

where:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \qquad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1. \tag{A1d}$$

We then recall that the boost matrices are expressed exponentially in eq. (40b) as:

$$b_{\kappa}\left(\beta\right) = \exp(iK^{\kappa}\xi), \tag{A2a}$$

and since $\beta = \tanh\xi$, changing ξ to $-\xi$ also changes β to $-\beta$, so the inverse boost matrix is:

$$b_{\kappa}(-\beta) = \exp(-iK^{\kappa}\xi). \tag{A2b}$$

We also recall that the electromagnetic transformation matrices are expressed exponentially in eq. (41) as:

$$h_{\kappa}(\beta) = \exp(-J^{\kappa}\xi). \tag{A2c}$$

These equations are next used in Appendix C.

APPENDIX B: EXPAND $EXP(G^{\kappa}\xi)$ IN A SERIES EXPANSION TO SHOW THAT $EXP(G^{\kappa}\xi) = G^0 COSH\xi + G^{\kappa} SINH\xi$.

We expand $\exp(G^{\kappa}\xi)$ in the series:

$$\exp(G^{\kappa}\xi) = \sum_{n=0}^{\infty}(G^{\kappa}\xi)^n = \sum_{n=0,2,4,\ldots}(G^{\kappa})^n\xi^n + \sum_{n=1,3,5,\ldots}\left(G^{\kappa}\right)^n\xi^n, \qquad \text{(B1)}$$

and recall eq. (A1b) as $\left(G^{\kappa}\right)^2 = G^0 = I$ to find that:

$$\exp\left(G^{\kappa}\xi\right) = G^0\sum_{n=0,2,4,\ldots}\xi^n + G^{\kappa}\sum_{n=1,3,5,\ldots}\xi^n = G^0\cosh\xi + G^{\kappa}\sinh\xi = G^0\gamma + G^{\kappa}\beta\gamma. \qquad \text{(B2)}$$

APPENDIX C: USE APPENDICES A AND B TO SHOW THAT $B_{\kappa}(-\beta)G^0 H_{\kappa}(\beta) = G^0 COSH\xi - G^{\kappa} SINH\xi$.

Recalling eq. (A2), we obtain:

$$b_{\kappa}(-\beta)G^0 h_{\kappa}(\beta) = \exp\left(-iK^{\kappa}\xi\right)\exp\left(-J^{\kappa}\xi\right) = \exp\left[\left(J^{\kappa}+iK^{\kappa}\right)(-\xi)\right] = \exp\left[G^{\kappa}(-\xi)\right]. \qquad \text{(C1)}$$

Replacing ξ by -ξ in eq. (B2), we deduce:

$$G^{\kappa}(-\xi) = G^0\cosh(-\xi) + G^{\kappa}\sinh(-\xi) = G^0\cosh\xi - G^{\kappa}\sinh\xi, \qquad \text{(C2)}$$

so tha:t

$$b_{\kappa}(-\beta)G^0 h_{\kappa}(\beta) = G^0\cosh\xi - G^{\kappa}\sinh\xi = G^0\gamma - G^{\kappa}\beta\gamma. \qquad \text{(C3)}$$

APPENDIX D: USE APPENDICES A, B, AND C TO SHOW THAT $B_{\kappa}(-\beta)G^{\kappa} H_{\kappa}(\beta) = G^{\kappa} COSH\xi - G^0 SINH\xi$.

Recalling eq. (A2), we obtain:

$$b_{\kappa}(-\beta)G^{\kappa} h_{\kappa}(\beta) = \exp(-iK^{\kappa}\xi)G^{\kappa}\exp\left(-J^{\kappa}\xi\right) = \exp\left(-iK^{\kappa}\xi\right)\left(J^{\kappa}+iK^{\kappa}\right)\exp\left(-J^{\kappa}\xi\right). \qquad \text{(D1)}$$

The matrix $\exp(-K^{\kappa}\xi)$ is an expansion in K^{κ} so $\exp\left(-K^{\kappa}\xi\right)$ commutes with K^{κ}. Also K^{κ} commutes with J^{κ}, so $\exp\left(-K^{\kappa}\xi\right)$ also commutes with J^{κ}. These two commutations tell us that:

$$\exp\left(-K^{\kappa}\xi\right)G^{\kappa} = G^{\kappa}\exp\left(-K^{\kappa}\xi\right), \tag{D2}$$

so:

$$b_{\kappa}\left(-\beta\right)G^{\kappa}h_{\kappa}\left(\beta\right) = G^{\kappa}\exp\left(-iK^{\kappa}\xi\right)\exp\left(-J^{\kappa}\xi\right) = G^{\kappa}\exp\left[G^{\kappa}\left(-\xi\right)\right], \tag{D3}$$

and then recalling eq. (C2), we obtain:

$$b_{\kappa}\left(-\beta\right)G^{\kappa}h_{\kappa}\left(\beta\right) = G^{\kappa}\left[G^{0}\cosh\xi - G^{\kappa}\sinh\xi\right] = G^{\kappa}\cosh\xi - G^{0}\sinh\xi = G^{\kappa}\gamma - G^{0}\beta\gamma. \tag{D4}$$

APPENDIX E: USE EQ.(B2), $EXP(G^{\kappa}\xi) = G^{0}COSH\xi + G^{\kappa}SINH\xi$, TO SHOW THAT $EXP\left(G^{\kappa}\xi/2\right)G^{\mu} = G^{\mu}EXP\left(-G^{\kappa}\xi/2\right)$ WHEN $\mu \neq k$.

Changing ξ to ξ/2 in eq. (B2) and multiplying times G^{μ} gives:

$$\begin{aligned}\exp\left(G^{\kappa}\xi/2\right)G^{\mu} &= \left[G^{0}\cosh\left(\xi/2\right) + G^{\kappa}\sinh\left(\xi/2\right)\right]G^{\mu} \\ &= G^{\mu}\cosh\left(\xi/2\right) + G^{\kappa}G^{\mu}\sinh(\xi/2).\end{aligned} \tag{E1}$$

Then, changing ξ to -ξ/2 eq. (B2) and multiplying by G^{μ} gives:

$$\begin{aligned}G^{\mu}\exp(-G^{\kappa}\xi/2) &= G^{\mu}\left[G^{0}\cosh(\xi/2) - G^{\kappa}\sinh(\xi/2)\right] \\ &= G^{\mu}\cosh(\xi/2) - G^{\mu}G^{\kappa}\sinh(\xi/2).\end{aligned} \tag{E2}$$

We now compare eq. (E1) with eq. (E2), and recalling eq. (A1c) $G^{\mu}G^{\kappa} = -G^{\kappa}G^{\mu}$

when μ, k = 1, 2, 3 and $\mu \neq k$, we find that:

$$\exp\left(G^{\kappa}\xi/2\right)G^{\mu} = G^{\mu}\exp\left(-G^{\kappa}\xi/2\right) \ \text{ when } \ \mu, k = 1, 2, 3 \ \text{ and } \ \mu \neq k. \tag{E3}$$

APPENDIX F: FOR μ ≠ k, INTRODUCE THE MATRIX $P^{\kappa} = J^{\kappa} - IK^{\kappa}$, AND THEN USE $EXP\left(G^{\kappa}\xi/2\right)$ AND $EXP\left(-P^{\kappa}\xi/2\right)$ WITH EQ.(E3) TO SHOW THAT $EXP\left(IK^{\kappa}\xi\right)G^{\mu} = G^{\mu}EXP\left[\left(-J^{\kappa}\xi\right)\right]$.

We define a complementary generator matrix P^{κ} as:

$$P^{\kappa} = J^{\kappa} - iK^{\kappa}, \tag{F1}$$

so:

$$\exp(iK^{\kappa}\xi) = \exp\left[\left(iK^{\kappa} + J^{\kappa}\right)\xi/2\right]\exp\left[\left(-J^{\kappa} + iK^{\kappa}\right)\xi/2\right]$$
$$= \exp\left[G^{\kappa}\xi/2\right]\exp\left[-P^{\kappa}\xi/2\right], \tag{F2}$$

and:

$$\exp\left(iK^{\kappa}\xi\right)G^{\mu} = \exp\left[G^{\kappa}\xi/2\right]\exp\left[-P^{\kappa}\xi/2\right]G^{\mu}. \tag{F3}$$

We now recall Ryder´s discussion [16] showing that the matrices G^{κ} and P^{κ} always commute (for k, μ = 1, 2, 3), so $\exp\left(-P_{\kappa}\xi/2\right)$ also commutes with G_{μ}, and we obtain:

$$\exp\left(iK^{\kappa}\xi\right)G^{\mu} = \exp\left[G^{\kappa}\xi/2\right]G^{\mu}\exp\left[-P^{\kappa}\xi/2\right]. \tag{F4}$$

We then recall eq.(E3) to rewrite this expression as:

$$\exp\left(iK^{\kappa}\xi\right)G^{\mu} = G^{\mu}\exp\left[-G^{\kappa}\xi/2\right]\exp\left[-P^{\kappa}\xi/2\right]$$
$$= G^{\mu}\exp\left[\left(-J^{\kappa} - iK^{\kappa}\right)\xi/2\right]\exp\left[\left(iK^{\kappa} - J^{\kappa}\right)\xi/2\right] = G^{\mu}\exp\left[\left(-J^{\kappa}\xi\right)\right], \tag{F5}$$

but recalling eq. (A2) we can rewrite this result as:

$$G^{\mu}h_{\kappa}\left(\beta\right) = b_{\kappa}\left(\beta\right)G^{\mu}, \tag{F6}$$

and this expression immediately demonstrates that:

$$b_{\kappa}\left(-\beta\right)G^{\mu}h_{\kappa}\left(\beta\right) = b_{\kappa}\left(-\beta\right)b_{\kappa}\left(\beta\right)G^{\mu} = G^{\mu}, \text{ when } \mu, k = 1, 2, 3 \text{ and } \mu \neq k. \tag{F7}$$

This equation completes our Lorentz group algebra development since equations C3, D4 and F7 can be used to demonstrate Lorentz invariance of Maxwell-Lorentz matrix expression of Maxwell equations.