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Newman-Penrose's formalism

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ABSTRACT

We obtain the Newman-Penrose (NP) version of the Riemann tensor and the corresponding Bianchi identities. We also realize the NP transcription of the Weyl-Lanczos equations, and we emphasize that it is an open problem to find general solutions of this equations for the Petrov types I, II, and D, and to establish the possible physical meaning of the Lanczos potential.

***Keywords*:** Identities of Bianchi, Newman-Penrose equations, Weyl tensor, Lanczos generator

1. INTRODUCTION

We shall employ the notation and conventions of [1-3]. The Riemann tensor, in terms of the Christoffel symbols, is given by:

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\alpha,\beta} - \Gamma^{\mu}{}_{\nu\beta,\alpha} + \Gamma^{\lambda}{}_{\nu\alpha}\Gamma^{\mu}{}_{\lambda\beta} - \Gamma^{\lambda}{}_{\nu\beta}\Gamma^{\mu}{}_{\lambda\alpha}\,, \tag{1}$$

that is:

$$V_{\mu;\nu\alpha} - V_{\mu;\alpha\nu} = R^{\beta}{}_{\mu\alpha\nu}V_{\beta}\,, \tag{2}$$

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for an arbitrary vectorial field V_α. Thus $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = R^{\mu}{}_{\mu}$ are the Ricci tensor and scalar curvature, respectively. The totally antisymmetric tensors of Levi-Civita:

$$\eta^{\mu\nu\alpha\beta} = -\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}, \qquad \eta_{\mu\nu\alpha\beta} = \sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta}, \qquad g = \det(g_{ab}), \tag{3}$$

permit to define the double dual [4, 5]:

$$^{*}R^{*ij}{}_{km} = \frac{1}{4}\eta^{ij\mu\nu}\,\eta_{km\alpha\beta}\,R_{\mu\nu}{}^{\alpha\beta}, \qquad ^{*}R^{*\mu}{}_{\alpha\mu\beta} = G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta}, \tag{4}$$

with the same algebraic symmetries as Riemann tensor.

If into Bianchi identities [6-8]:

$$R_{\mu\nu\alpha\beta;\lambda} + R_{\mu\nu\beta\lambda;\alpha} + R_{\mu\nu\lambda\alpha;\beta} = 0, \qquad G_{\alpha}{}^{\beta}{}_{;\beta} = 0, \tag{5}$$

is used the Einstein [9]-Lanczos [10, 11] relation:

$$^{*}R^{*}{}_{ijkm} = -R_{ijkm} + R_{ik}g_{jm} + R_{jm}g_{ik} - R_{im}g_{jk} - R_{jk}g_{im} + \frac{R}{2}\left(g_{im}g_{jk} - g_{ik}g_{jm}\right), \tag{6}$$

then we deduce the differential property [8, 11]:

$$^{*}R^{*\mu}{}_{\nu\alpha\beta;\mu} = 0. \tag{7}$$

On the other hand, Debever-Géhéniau [12] obtained the irreducible decomposition:

$$R_{imnj} = C_{imnj} + \frac{1}{2}\left(E_{in}g_{mj} + E_{mj}g_{in} - E_{ij}g_{mn} - E_{mn}g_{ij}\right) + \frac{R}{12}\left(g_{in}g_{mj} - g_{ij}g_{mn}\right), \tag{8}$$

where $E_{\mu\nu} = R_{\mu\nu} - \frac{R}{4}g_{\mu\nu}$ and $C_{\mu\nu\alpha\beta}$ is the conformal tensor [1, 7, 8]; therefore, (5),…, (8) imply the following expression for the Bianchi identities:

$$C^{\mu}{}_{\nu\alpha\beta;\mu} = \frac{1}{2}\left(E_{\nu\beta;\alpha} - E_{\nu\alpha;\beta}\right) + \frac{1}{24}\left(R_{,\alpha}\,g_{\nu\beta} - R_{,\beta}\,g_{\nu\alpha}\right). \tag{9}$$

In Sec. 2 we deduce the Newman-Penrose (NP) [13, 14] versions of (5) and (8); we also exhibit the spinor transcription of (9).

The work [1] contains an algebraic study of the Lanczos potential [11, 15-19], now here we consider differential aspects to write the Weyl tensor in terms of its generator:

$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + K_{\mu\beta}g_{\nu\alpha} - K_{\mu\alpha}g_{\nu\beta} + K_{\nu\alpha}g_{\mu\beta} - K_{\nu\beta}g_{\mu\alpha}, \tag{10}$$

where:

$$K_{\mu\nu} = K_{\nu\mu} = K_{\mu}{}^{\alpha}{}_{\nu;\alpha}, \tag{11}$$

with the properties [1]:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}\,, \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0\,, \qquad K_{\mu\ \ \nu}^{\ \nu} = 0\,, \tag{12}$$

and the divergence-free gauge condition:

$$K_{\mu\nu}{}^{\alpha}{}_{;\alpha} = 0\,. \tag{13}$$

We remember the importance of the dual spintensor:

$${}^{*}K_{\mu\nu\alpha} = -\,{}^{*}K_{\nu\mu\alpha} = \tfrac{1}{2}\eta_{\mu\nu pq}K^{pq}{}_{\alpha}\,, \qquad {}^{*}K_{\mu\nu}{}^{\alpha}{}_{;\alpha} = 0\,, \qquad {}^{*}K_{\mu}{}^{\nu}{}_{\nu} = 0\,, \tag{14}$$

then from (10):

$${}^{*}C_{\mu\nu\alpha\beta} = {}^{*}K_{\mu\nu\alpha;\beta} - {}^{*}K_{\mu\nu\beta;\alpha} + {}^{*}K_{\alpha\beta\mu;\nu} - {}^{*}K_{\alpha\beta\nu;\mu} + {}^{*}K_{\mu\beta}g_{\nu\alpha} - {}^{*}K_{\mu\alpha}g_{\nu\beta} + {}^{*}K_{\nu\alpha}g_{\mu\beta} - {}^{*}K_{\nu\beta}g_{\mu\alpha}\,, \tag{15}$$

where: ${}^{*}K_{\mu\nu} = {}^{*}K_{\nu\mu} = {}^{*}K_{\mu}{}^{\alpha}{}_{\nu;\alpha}$; hence ${}^{*}K_{\mu\nu\alpha}$ is the generator of ${}^{*}C_{\mu\nu\alpha\beta}$. Thus, it is natural to construct a complex expression equivalent to (10) and (15):

$$S_{\mu\nu\alpha\beta} = S_{\mu\nu\alpha;\beta} - S_{\mu\nu\beta;\alpha} + S_{\alpha\beta\mu;\nu} - S_{\alpha\beta\nu;\mu} + H_{\mu\beta}g_{\nu\alpha} - H_{\mu\alpha}g_{\nu\beta} + H_{\nu\alpha}g_{\mu\beta} - H_{\nu\beta}g_{\mu\alpha},$$
$$S_{\mu\nu}{}^{\alpha}{}_{;\alpha} = 0, \tag{16}$$

such that:

$$S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i\,{}^{*}C_{\mu\nu\alpha\beta} \qquad \text{Complex Weyl tensor,}$$

$$S_{\mu\nu\alpha} = K_{\mu\nu\alpha} + i\,{}^{*}K_{\mu\nu\alpha} \qquad \text{Complex Lanczos potential,} \tag{17}$$

$$S_{\mu}{}^{\nu}{}_{\nu} = 0\,, \qquad H_{\mu\nu} = S_{\mu}{}^{\alpha}{}_{\nu;\alpha} = K_{\mu\nu} + i\,{}^{*}K_{\mu\nu}\,.$$

In Sec. 3 we obtain the spinor and NP versions of (16).

2. RIEMANN TENSOR AND BIANCHI IDENTITIES IN THE NP FORMALISM

It is simple to project (2) (for $V_{\mu} = Z_{(a)\mu}$) and (8) onto null tetrad:

$$R_{(a)(b)(c)(d)} = C_{(a)(b)(c)(d)} + \tfrac{1}{2}\big(E_{(a)(d)}Z_{(b)(c)} + E_{(b)(c)}Z_{(a)(d)} - E_{(a)(c)}Z_{(b)(d)} - E_{(b)(d)}Z_{(a)(c)}\big) + \tfrac{R}{12}\big(Z_{(a)(d)}Z_{(b)(c)} - Z_{(a)(c)}Z_{(b)(d)}\big),$$
$$= \gamma_{abc;k}Z_{(d)}{}^{k} - \gamma_{abd;k}Z_{(c)}{}^{k} + \big(\gamma^{k}{}_{dc} - \gamma^{k}{}_{cd}\big)\gamma_{abk} + \big(\gamma^{k}{}_{bc}\gamma_{akd} - \gamma^{k}{}_{bd}\gamma_{akc}\big), \tag{18}$$

with the notation $\gamma^{a}{}_{bc} = Z^{(a)(r)}\gamma_{rbc}$, and for adequate values of the indices we find the NP version of curvature tensor [8, 13, 14, 20-23]:

Newman-Penrose equations

$$
\begin{aligned}
&\text{(a)}\ \ D\rho-\bar{\delta}\kappa=(\rho+\varepsilon+\bar{\varepsilon})\rho+\sigma\bar{\sigma}-\bar{\kappa}\tau-\left(3\alpha+\bar{\beta}-\pi\right)\kappa-\phi_{00}\,,\\
&\text{(b)}\ \ D\sigma-\delta\kappa=(\rho+\bar{\rho}+3\varepsilon-\bar{\varepsilon})\sigma-(\tau-\bar{\pi}+\bar{\alpha}+3\beta)\kappa+\psi_0\,,\\
&\text{(c)}\ \ D\tau-\Delta\kappa=(\tau+\bar{\pi})\rho+(\bar{\tau}+\pi)\sigma+(\varepsilon-\bar{\varepsilon})\tau-(3\gamma+\bar{\gamma})\kappa+\psi_1-\phi_{01}\,,\\
&\text{(d)}\ \ D\alpha-\bar{\delta}\varepsilon=(\varrho+\bar{\varepsilon}-2\varepsilon)\alpha+\beta\bar{\sigma}-\bar{\beta}\varepsilon-\kappa\lambda-\bar{\kappa}\gamma+(\varepsilon+\rho)\pi-\phi_{10}\,,\\
&\text{(e)}\ \ D\beta-\delta\varepsilon\ =(\alpha+\pi)\sigma+(\bar{\rho}-\bar{\varepsilon})\beta-(\mu+\gamma)\kappa-(\bar{\alpha}-\bar{\pi})\varepsilon+\psi_1\,,\\
&\text{(f)}\ \ D\gamma-\Delta\varepsilon\ =(\tau+\bar{\pi})\alpha+(\bar{\tau}+\pi)\beta-(\varepsilon+\bar{\varepsilon})\gamma-(\gamma+\bar{\gamma})\varepsilon+\tau\pi-\nu\kappa+\psi_2-\phi_{11}+\frac{R}{24}\,,\\
&\text{(g)}\ \ D\lambda-\bar{\delta}\pi=(\rho-3\varepsilon+\bar{\varepsilon})\lambda+\left(\pi+\alpha-\bar{\beta}\right)\pi+\bar{\sigma}\mu-\nu\bar{\kappa}-\phi_{20}\,,\\
&\text{(h)}\ \ D\mu-\delta\pi=(\bar{\rho}-\varepsilon-\bar{\varepsilon})\mu+(\bar{\pi}-\bar{\alpha}+\beta)\pi+\sigma\lambda-\nu\kappa+\psi_2-\frac{R}{12}\,,\\
&\text{(i)}\ \ D\nu-\Delta\pi\ =(\mu+\gamma-\bar{\gamma})\pi+\bar{\tau}\mu+(\bar{\pi}+\tau)\lambda-(3\varepsilon+\bar{\varepsilon})\nu+\psi_3-\phi_{21}\,,\\
&\text{(j)}\ \ \Delta\lambda-\bar{\delta}\nu\ =(\bar{\gamma}-3\gamma-\mu-\bar{\mu})\lambda+\left(3\alpha+\bar{\beta}+\pi-\bar{\tau}\right)\nu-\psi_4\,,\\
&\text{(k)}\ \ \delta\rho-\bar{\delta}\sigma=(\bar{\alpha}+\beta)\rho-\left(3\alpha-\bar{\beta}\right)\sigma+(\rho-\bar{\rho})\tau+(\mu-\bar{\mu})\kappa-\psi_1-\phi_{01}\,,\\
&\text{(l)}\ \ \delta\alpha-\bar{\delta}\beta\ =(\rho+\varepsilon)\mu-\lambda\sigma+\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta+(\rho-\bar{\varrho})\gamma-\varepsilon\bar{\mu}-\psi_2-\phi_{11}-\frac{R}{24}\,,\\
&\text{(m)}\ \ \delta\lambda-\bar{\delta}\mu=(\rho-\bar{\rho})\nu+\left(\alpha+\bar{\beta}+\pi\right)\mu-\pi\bar{\mu}+(\bar{\alpha}-3\beta)\lambda-\psi_3-\phi_{21}\,,\\
&\text{(n)}\ \ \delta\nu-\Delta\mu\ =(\mu+\gamma+\bar{\gamma})\mu+\lambda\bar{\lambda}-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu-\phi_{22}\,,\\
&\text{(o)}\ \ \delta\gamma-\Delta\beta\ =(\tau-\bar{\alpha}-2\beta)\gamma+(\tau+\beta)\mu-\sigma\nu-\varepsilon\bar{\nu}+\beta\bar{\gamma}+\alpha\bar{\lambda}-\phi_{12}\,,\\
&\text{(p)}\ \ \delta\tau-\Delta\sigma\ =(\mu-3\gamma+\bar{\gamma})\sigma+\bar{\lambda}\rho+(\tau+\beta-\bar{\alpha})\tau-\kappa\bar{\nu}-\phi_{02}\,,\\
&\text{(q)}\ \ \Delta\rho-\bar{\delta}\tau\ =(\gamma+\bar{\gamma}-\bar{\mu})\rho-\sigma\lambda+\left(\bar{\beta}-\alpha-\bar{\tau}\right)\tau+\nu\kappa-\psi_2+\frac{R}{12}\,,\\
&\text{(r)}\ \ \Delta\alpha-\bar{\delta}\gamma\ =(\rho+\varepsilon)\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+\left(\bar{\beta}-\bar{\tau}\right)\gamma-\psi_3\,,
\end{aligned}
\tag{19}
$$

with the following association between the components (18) and each equation in (19):

$$
\begin{aligned}
&\text{(a): } R_{(3)(1)(1)(4)}\,, && \text{(j): } R_{(4)(2)(4)(2)}\,, && \text{(d): } R_{(1)(2)(4)(1)}-R_{(3)(4)(4)(1)},\\
&\text{(b): } R_{(3)(1)(3)(1)}\,, && \text{(k): } R_{(3)(1)(4)(3)}\,, && \text{(e): } R_{(1)(2)(3)(1)}+R_{(4)(3)(3)(1)},\\
&\text{(c): } R_{(3)(1)(1)(2)}\,, && \text{(m): } R_{(4)(2)(4)(3)}\,, && \text{(f): } R_{(1)(2)(1)(2)}-R_{(3)(4)(1)(2)},\\
&\text{(g): } R_{(4)(2)(1)(4)}\,, && \text{(n): } R_{(4)(2)(3)(2)}\,, && \text{(l): } R_{(3)(4)(3)(4)}-R_{(1)(2)(3)(4)},\\
&\text{(h): } R_{(4)(2)(3)(1)}\,, && \text{(p): } R_{(3)(1)(3)(2)}\,, && \text{(o): } R_{(3)(4)(3)(2)}-R_{(1)(2)(3)(2)},\\
&\text{(i): } R_{(4)(2)(1)(2)}\,, && \text{(q): } R_{(3)(1)(4)(2)}\,, && \text{(r): } R_{(4)(3)(4)(2)}+R_{(2)(2)(4)(2)}.
\end{aligned}
\tag{20}
$$

The NP equations (19) are equivalent to the commutators (to remember the equations (5) in [2]):

$$
\begin{aligned}
[D,\Delta]o^A=\ &[(\tau+\bar{\pi})\alpha+(\bar{\tau}+\pi)\beta-(\varepsilon+\bar{\varepsilon})\gamma-(\gamma+\bar{\gamma})\varepsilon+\psi_2-\phi_{11}+\frac{R}{24}]o^A-\\
&-[(\tau+\bar{\pi})\rho+(\bar{\tau}+\pi)\sigma-(\varepsilon+\bar{\varepsilon})\tau-(\gamma+\bar{\gamma})\kappa+\psi_1-\phi_{01}]\iota^A,
\end{aligned}
$$

$$
[D,\Delta]\iota^A=\ [(\tau+\bar{\pi})\lambda+(\bar{\tau}+\pi)\mu-(\varepsilon+\bar{\varepsilon})\nu-(\gamma+\bar{\gamma})\pi+\psi_3-\phi_{21}]o^A+(\iota_B[D,\Delta]o^B)\iota^A,
$$

$$[D,\delta]o^A = [\alpha\sigma + (\bar{\rho}-\bar{\varepsilon})\beta - \kappa\gamma - (\bar{\alpha}-\bar{\pi})\varepsilon + \psi_1]o^A + [(\tau - \bar{\pi} + \bar{\alpha} + \beta)\kappa - (\rho + \bar{\rho} + \varepsilon + \bar{\varepsilon})\sigma - \psi_0]\iota^A,$$

$$[D,\delta]\iota^A = [(\bar{\rho} - 3\varepsilon - \bar{\varepsilon})\mu + (\bar{\pi} - \beta - \bar{\alpha})\pi - \nu\kappa + \psi_2 - \frac{R}{12}]o^A + (\iota_B[D,\delta]o^B)\iota^A,$$

$$[D,\bar{\delta}]o^A = [(\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} + (\pi - \bar{\beta})\varepsilon - \bar{\kappa}\gamma - \phi_{10}]o^A + [(\varepsilon - \bar{\varepsilon} - \rho)\rho - \sigma\bar{\sigma} + \bar{\kappa}\tau + (\alpha + \bar{\beta} - \pi)\kappa + \phi_{00}]\iota^A,$$

$$[D,\bar{\delta}]\iota^A = [\rho\lambda + \bar{\sigma}\mu + (\pi - \alpha - \bar{\beta})\pi - \nu\bar{\kappa} + (\bar{\varepsilon} - \varepsilon)\lambda - \phi_{20}]o^A + (\iota_B[D,\bar{\delta}]o^B)\iota^A, \tag{21}$$

$$[\delta,\bar{\delta}]o^A = \left[\alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\varepsilon - \psi_2 - \phi_{11} - \frac{R}{24}\right]o^A + [(\beta - \bar{\alpha})\rho + (\alpha - \bar{\beta})\sigma + (\bar{\rho} - \rho)\tau + (\bar{\mu} - \mu)\kappa + \psi_1 + \phi_{01}]\iota^A,$$

$$[\delta,\bar{\delta}]\iota^A = [(\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\bar{\beta} - \alpha)\mu + (\bar{\alpha} - \beta)\lambda - \psi_3 - \phi_{21}]o^A + (\iota_B[\delta,\bar{\delta}]o^B)\iota^A,$$

$$[\Delta,\delta]o^A = [(\bar{\alpha} + \beta - \tau)\gamma + \varepsilon\bar{\nu} + (\gamma - \bar{\gamma} - \mu)\beta - \alpha\bar{\lambda} + \phi_{12}]o^A + [(\mu + \bar{\gamma} - \gamma)\sigma + (\tau - \beta - \bar{\alpha})\tau + \bar{\lambda}\rho - \kappa\bar{\nu} - \phi_{02}]\iota^A,$$

$$[\Delta,\delta]\iota^A = [(\gamma - \bar{\gamma} - \mu)\mu - \lambda\bar{\lambda} + \bar{\nu}\pi + (\bar{\alpha} - \tau + \beta)\nu + \phi_{22}]o^A + (\iota_B[\Delta,\delta]o^B)\iota^A,$$

$$[\Delta,\bar{\delta}]o^A = [\varepsilon\nu - \beta\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \psi_3]o^A + \left[(\bar{\mu} - \bar{\gamma} + \gamma)\rho + \sigma\lambda - \nu\kappa + (\bar{\tau} - \alpha - \bar{\beta})\tau + \psi_2 - \frac{R}{12}\right]\iota^A,$$

$$[\Delta,\bar{\delta}]\iota^A = [(\bar{\gamma} - \gamma - \mu - \bar{\mu})\lambda + (\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \psi_4]o^A + (\iota_B[\Delta,\bar{\delta}]o^B)\iota^A.$$

Similarly, the projection of (5) onto null tetrad is given by:

$$Z_{(a)(b)(c)(d)(e)} \equiv R_{(a)(b)(c)(d);r}\, Z_{(e)}{}^r + R_{(a)(b)(d)(e);r}\, Z_{(c)}{}^r + R_{(a)(b)(e)(c);r}\, Z_{(d)}{}^r + \gamma^m{}_{be}R_{(m)(a)(c)(d)} + \gamma^m{}_{bc}R_{(m)(a)(d)(e)} + \gamma^m{}_{bd}R_{(m)(a)(e)(c)} - \gamma^m{}_{ae}R_{(m)(b)(c)(d)} - \gamma^m{}_{ac}R_{(m)(b)(d)(e)} - \gamma^m{}_{ad}R_{(m)(b)(e)(c)} + (\gamma^m{}_{ec} - \gamma^m{}_{ce})R_{(m)(d)(a)(b)} + (\gamma^m{}_{de} - \gamma^m{}_{ed})R_{(m)(c)(a)(b)} + (\gamma^m{}_{cd} - \gamma^m{}_{dc})R_{(m)(e)(a)(b)} = 0\,, \tag{22}$$

and for convenient values of the indices we obtain [8, 13, 14, 20-23]:

<u>Bianchi identities</u>

(a) $\bar{\delta}\psi_0 - D\psi_1 + \delta\phi_{00} - D\phi_{01} = (4\alpha - \pi)\psi_0 - 2(2\rho + \varepsilon)\psi_1 + 3\kappa\psi_2 + (2\bar{\alpha} + 2\beta - \bar{\pi})\phi_{00} - 2(\varepsilon + \bar{\rho})\phi_{01} + \bar{\kappa}\phi_{02} - 2\sigma\phi_{10} + 2\kappa\phi_{11}\,,$

(b) $\Delta\psi_0 - \delta\psi_1 + \delta\phi_{01} - D\phi_{02} = (4\gamma - \mu)\psi_0 - 2(2\tau + \beta)\psi_1 + 3\sigma\psi_2 + \bar{\lambda}\phi_{00} + 2(\beta - \bar{\pi})\phi_{01} - (2\varepsilon - 2\bar{\varepsilon} + \bar{\rho})\phi_{02} - 2\sigma\phi_{11} + 2\kappa\phi_{12}\,,$

(c) $\bar{\delta}\psi_3 - D\psi_4 + \Delta\phi_{20} - \bar{\delta}\phi_{21} = 3\lambda\psi_2 - 2(2\pi + \alpha)\psi_3 + (4\varepsilon - \rho)\psi_4 + 2\nu\phi_{10} - 2\lambda\phi_{11}$ $-(2\gamma - 2\bar{\gamma} + \bar{\mu})\phi_{20} + 2(\alpha - \bar{\tau})\phi_{21} + \bar{\sigma}\phi_{22}$,

(d) $\Delta\psi_3 - \delta\psi_4 + \Delta\phi_{21} - \bar{\delta}\phi_{22} = 3\nu\psi_2 - 2(2\mu + \gamma)\psi_3 + (4\beta - \tau)\psi_4 + 2\nu\phi_{11} - 2\lambda\phi_{12} +$ $\bar{\nu}\phi_{20} - 2(\gamma + \bar{\mu})\phi_{21} + (2\bar{\beta} + 2\alpha - \bar{\tau})\phi_{22}$,

(e) $\bar{\delta}\psi_1 - D\psi_2 + \Delta\phi_{00} - \bar{\delta}\phi_{01} + \frac{1}{12}DR = \lambda\psi_0 + 2(\alpha - \pi)\psi_1 - 3\rho\psi_2 + 2\kappa\psi_3 +$ $(2\gamma + 2\bar{\gamma} - \bar{\mu})\phi_{00} - 2(\bar{\tau} + \alpha)\phi_{01} + \bar{\sigma}\phi_{02} - 2\tau\phi_{10} + 2\rho\phi_{11}$,

(23)

(f) $\Delta\psi_2 - \delta\psi_3 + \delta\phi_{21} - D\phi_{22} - \frac{1}{12}\Delta R = 2\nu\psi_1 - 3\mu\psi_2 + 2(\beta - \tau)\psi_3 + \sigma\psi_4 + 2\mu\phi_{11} -$ $2\pi\phi_{12} + \bar{\lambda}\phi_{20} - 2(\bar{\pi} + \beta)\phi_{21} + (2\varepsilon + 2\bar{\varepsilon} - \bar{\rho})\phi_{22}$,

(g) $\bar{\delta}\psi_2 - D\psi_3 + \delta\phi_{20} - D\phi_{21} - \frac{1}{12}\bar{\delta}R = 2\lambda\psi_1 - 3\pi\psi_2 + 2(\varepsilon - \rho)\psi_3 + \kappa\psi_4 + 2\mu\phi_{10} -$ $2\pi\phi_{11} - (2\beta + \bar{\pi} - 2\bar{\alpha})\phi_{20} + 2(\varepsilon - \bar{\rho})\phi_{21} + \bar{\kappa}\phi_{22}$,

(h) $\Delta\psi_1 - \delta\psi_2 + \Delta\phi_{01} - \bar{\delta}\phi_{02} + \frac{1}{12}\delta R = \nu\psi_0 + 2(\gamma - \mu)\psi_1 - 3\tau\psi_2 + 2\sigma\psi_3 + \bar{\nu}\phi_{00} +$ $2(\gamma - \bar{\mu})\phi_{01} - (\bar{\tau} - 2\bar{\beta} + 2\alpha)\phi_{02} - 2\tau\phi_{11} + 2\rho\phi_{12}$,

(i) $\Delta\phi_{11} - \bar{\delta}\phi_{12} - \delta\phi_{21} + D\phi_{22} + \frac{1}{8}\Delta R = \nu\phi_{01} - \lambda\phi_{02} + \bar{\nu}\phi_{10} - 2(\mu + \bar{\mu})\phi_{11} +$ $(2\bar{\beta} + 2\pi - \bar{\tau})\phi_{12} - \bar{\lambda}\phi_{20} + (2\beta + 2\bar{\pi} - \tau)\phi_{21} + (\rho + \bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\phi_{22}$,

(j) $\Delta\phi_{00} - \bar{\delta}\phi_{01} - \delta\phi_{10} + D\phi_{11} + \frac{1}{8}DR = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\phi_{01} +$ $\bar{\sigma}\phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\phi_{10} + 2(\rho + \bar{\rho})\phi_{11} - \bar{\kappa}\phi_{12} + \sigma\phi_{20} - \kappa\phi_{21}$,

(k) $\Delta\phi_{01} - \bar{\delta}\phi_{02} - \delta\phi_{11} + D\phi_{12} + \frac{1}{8}\delta R = \bar{\nu}\phi_{00} + (2\gamma - 2\bar{\mu} - \mu)\phi_{01} + (-2\alpha + 2\bar{\beta} + \pi -$ $\bar{\tau})\phi_{02} - \bar{\lambda}\phi_{10} + 2(\bar{\pi} - \tau)\phi_{11} + (\bar{\rho} + 2\rho - 2\bar{\varepsilon})\phi_{12} + \sigma\phi_{21} - \kappa\phi_{22}$,

with the following relationship between the components (22) and each equation in (23):

(a): $Z_{(1)(3)(1)(3)(4)}$, (b): $Z_{(1)(3)(1)(3)(2)}$, (c): $Z_{(2)(4)(4)(1)(2)}$, (d): $Z_{(2)(4)(2)(4)(3)}$,
(e): $Z_{(1)(3)(1)(2)(4)}$, (f): $Z_{(2)(4)(1)(2)(3)}$, (g): $Z_{(2)(4)(4)(1)(3)}$, (h): $Z_{(1)(3)(2)(3)(4)}$,
(i): $Z_{(3)(4)(3)(4)(2)}$ and (f) + ($\bar{\text{f}}$) , (j): $Z_{(3)(4)(3)(4)(1)}$ and (e) + ($\bar{\text{e}}$) , (24)
(k): $\frac{1}{2}[Z_{(1)(2)(1)(2)(3)} + Z_{(3)(4)(1)(2)(3)}]$ and ($\bar{\text{g}}$) .

In the calculations to deduce (23) are useful the expressions:

$$R_{(2)(3)(1)(2)} = -\bar{\psi}_3 + \phi_{12}\,, \quad R_{(2)(3)(2)(4)} = -\phi_{22}\,, \quad R_{(2)(3)(4)(1)} = \bar{\psi}_2 - \frac{R}{12}\,,$$
$$R_{(1)(4)(1)(2)} = \bar{\psi}_1 - \phi_{10}\,,$$
$$R_{(1)(4)(3)(1)} = \phi_{00}\,, \qquad R_{(2)(3)(2)(3)} = \bar{\psi}_4\,, \qquad R_{(3)(4)(4)(2)} = -\psi_3 - \phi_{21}\,,$$
$$R_{(1)(4)(4)(3)} = -\bar{\psi}_1 - \phi_{10}$$

$$R_{(1)(2)(1)(2)} = \psi_2 + \bar{\psi}_2 - 2\phi_{11} + \frac{R}{12}\,, \quad R_{(1)(4)(1)(4)} = \bar{\psi}_0\,, \quad R_{(1)(4)(2)(4)} = \phi_{20}\,, \tag{25}$$
$$R_{(3)(4)(1)(2)} = \bar{\psi}_2 - \psi_2\,,$$
$$R_{(3)(4)(3)(4)} = \psi_2 + \bar{\psi}_2 + 2\phi_{11} + \frac{R}{12}\,.$$

The relations (i), (j) and (k) of the set (23) are equivalent to $G_{\mu}{}^{\nu}{}_{;\nu} = 0$.

Now we shall determine the spinor version of Bianchi identities via (9), then it is immediate that:

$$\nabla^{A\dot{B}} C_{ACEG\dot{B}\dot{D}\dot{F}\dot{H}} = \frac{1}{2}(\nabla_{G\dot{H}} E_{CE\dot{D}\dot{F}} - \nabla_{E\dot{F}} E_{CG\dot{D}\dot{H}}) + \frac{1}{24}(\varepsilon_{CE}\varepsilon_{\dot{D}\dot{F}}\nabla_{G\dot{H}} R - \varepsilon_{CG}\varepsilon_{\dot{D}\dot{H}}\nabla_{E\dot{F}} R), \tag{26}$$

but [1]:

$$E_{AC\dot{B}\dot{D}} = -2\phi_{AC\dot{B}\dot{D}}\,, \qquad C_{ACEG\dot{B}\dot{D}\dot{F}\dot{H}} = \psi_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}} + \varepsilon_{AC}\varepsilon_{EG}\psi_{\dot{B}\dot{D}\dot{F}\dot{H}}\,, \tag{27}$$

hence (26) implies:

$$\varepsilon_{\dot{F}\dot{H}}\nabla^{A}{}_{\dot{D}}\psi_{ACEG} + \varepsilon_{EG}\nabla_{C}{}^{\dot{B}}\psi_{\dot{B}\dot{D}\dot{F}\dot{H}} + \frac{1}{24}(\varepsilon_{CG}\varepsilon_{\dot{D}\dot{H}}\nabla_{E\dot{F}} R - \varepsilon_{CE}\varepsilon_{\dot{D}\dot{F}}\nabla_{G\dot{H}} R) = \nabla_{E\dot{F}}\phi_{CG\dot{D}\dot{H}} - \nabla_{G\dot{H}}\phi_{CE\dot{D}\dot{F}}\,, \tag{28}$$

whose multiplication by $\varepsilon^{CE}\varepsilon^{\dot{H}\dot{F}}$ gives:

$$\nabla^{E\dot{F}}\phi_{GE\dot{F}\dot{D}} = -\frac{1}{8}\nabla_{G\dot{D}} R\,, \tag{29}$$

which is equivalent to $G^{\mu\nu}{}_{;\nu} = 0$. We must remember that $\psi_{A}{}^{C}{}_{CG} = 0$ and $\phi^{C}{}_{C\dot{D}\dot{F}} = 0$.

We multiply (28) by $\varepsilon^{\dot{H}\dot{D}}$:

$$\nabla^{A}{}_{\dot{F}}\psi_{ACEG} + \frac{1}{24}(2\varepsilon_{CG}\nabla_{E\dot{F}} R - \varepsilon_{CE}\nabla_{G\dot{F}} R) = -\nabla_{G}{}^{\dot{D}}\phi_{CE\dot{D}\dot{F}}\,,$$

and we symmetrize it in *CE* to obtain the spinor form of Bianchi identities [8, 23-25]:

$$\nabla^{A}{}_{\dot{F}}\psi_{ACEG} + \frac{1}{24}(\varepsilon_{CG}\varepsilon_{AE} + \varepsilon_{EG}\varepsilon_{AC})\nabla^{A}{}_{\dot{F}} R = -\nabla_{G}{}^{\dot{D}}\phi_{CE\dot{D}\dot{F}}\,, \tag{30}$$

therefore $\nabla^{A}{}_{\dot{F}}\psi_{ACEG} = 0$ in vacuum space ($R_{\mu\nu} = 0$).

3. WEYL-LANCZOS EQUATIONS

First we shall deduce the spinor version of the divergence-free gauge condition (16):

$$S_{\mu\nu\alpha}{}^{;\alpha} = 0 \qquad \therefore \qquad \nabla^{P\dot{Q}} S_{ACP\dot{B}\dot{D}\dot{Q}} = 2\nabla^{P\dot{Q}}\left(L_{ACP\dot{Q}}\varepsilon_{\dot{B}\dot{D}}\right) = 0\,,$$

and we multiply it by $\varepsilon^{\dot{B}\dot{D}}$ to obtain the constraint [23]:

$$\nabla^{P\dot{Q}} L_{ACP\dot{Q}} = 0 \,, \tag{31}$$

for the Lanczos spinor [1].

Similarly, if (16) is multiplied by $\sigma^{\mu}{}_{A\dot{B}}\sigma^{\nu}{}_{C\dot{D}}\sigma^{\alpha}{}_{E\dot{F}}\sigma^{\beta}{}_{G\dot{H}}$:

$$\begin{aligned}\psi_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}} &= (\nabla_{G\dot{H}}L_{ACE\dot{F}} - \nabla_{E\dot{F}}L_{ACG\dot{H}})\varepsilon_{\dot{B}\dot{D}} + (\nabla_{C\dot{D}}L_{EGA\dot{B}} - \nabla_{A\dot{B}}L_{EGC\dot{D}})\varepsilon_{\dot{F}\dot{H}} + \\ &+\nabla^{M}{}_{\dot{B}}(L_{AME\dot{F}}\varepsilon_{CG}\varepsilon_{\dot{D}\dot{H}} - L_{AMG\dot{H}}\varepsilon_{CE}\varepsilon_{\dot{D}\dot{F}}) + \nabla^{M}{}_{\dot{D}}(L_{CMG\dot{H}}\epsilon_{AE}\varepsilon_{\dot{B}\dot{F}} - L_{CME}\varepsilon_{AG}\varepsilon_{\dot{B}\dot{H}}),\end{aligned} \tag{32}$$

where were employed the relations:

$$S_{ACEG\dot{B}\dot{D}\dot{F}\dot{H}} = 2\psi_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}}\,, \quad S_{ACE\dot{B}\dot{D}\dot{F}} = 2L_{ACE\dot{F}}\varepsilon_{\dot{B}\dot{D}}\,, \quad H_{AG\dot{B}\dot{H}} = -2\nabla^{M}{}_{\dot{B}}L_{AMG\dot{H}}. \tag{33}$$

If we apply $\varepsilon^{\dot{B}\dot{D}}\varepsilon^{\dot{F}\dot{H}}$ to (32):

$$2\psi_{ACEG} = \nabla_{A}{}^{\dot{B}}L_{CEG\dot{B}} + \nabla_{C}{}^{\dot{B}}L_{AEG\dot{B}} + \nabla_{E}{}^{\dot{B}}L_{ACE\dot{B}} + \nabla_{G}{}^{\dot{B}}L_{ACE\dot{B}}\,, \tag{34}$$

because we use (31). But $\nabla_{A}{}^{\dot{B}}L_{CEG\dot{B}} = \nabla_{C}{}^{\dot{B}}L_{AEG\dot{B}}$, hence (34) acquires the structure [17, 23, 26-30]:

$$\psi_{ABCD} = 2\nabla_{D}{}^{\dot{E}}L_{ABC\dot{E}}\,, \tag{35}$$

and it is the spinor form of the Weyl-Lanczos equations with both gauge conditions imposed.

The NP version of (31) is obtained if we project $S_{\mu\nu}{}^{\alpha}{}_{;\alpha} = 0$ onto null tetrad:

$$N_{(a)(b)} \equiv S_{(a)(b)(r);\alpha}Z^{(r)\alpha} + S_{(a)(b)(r)}Z^{(r)\alpha}{}_{;\alpha} - \gamma^{q}{}_{br}S_{(a)(q)}{}^{(r)} - \gamma^{q}{}_{ar}S_{(q)(b)}{}^{(r)} = 0\,, \tag{36}$$

and for adequate values of the indices we deduce the equations:

$$\begin{aligned}
&\text{(a)}\ \ \Delta\Omega_2 - \delta\Omega_3 - \bar{\delta}\Omega_6 + D\Omega_7 - 2\nu\Omega_1 + (3\mu + \bar{\mu} + \gamma - \bar{\gamma})\Omega_2 + (\bar{\alpha} - 3\beta + \tau - \bar{\pi})\Omega_3 + \\
&2\lambda\Omega_5 + \left(-\alpha - \bar{\beta} + \bar{\tau} - 3\pi\right)\Omega_6 + (3\varepsilon + \bar{\varepsilon} - \rho - \bar{\rho})\Omega_7 = 0\,, \\
&\text{(b)}\ \ \Delta\Omega_0 - \delta\Omega_1 - \bar{\delta}\Omega_4 + D\Omega_5 + (\mu + \bar{\mu} - 3\gamma - \bar{\gamma})\Omega_0 + (\bar{\alpha} + \beta - \bar{\pi} + 3\tau)\Omega_1 - 2\sigma\Omega_2 + \\
&\left(3\alpha - \bar{\beta} - \pi + \bar{\tau}\right)\Omega_4 + (\bar{\varepsilon} - \varepsilon - \bar{\rho} - 3\rho)\Omega_5 + 2\kappa\Omega_6 = 0\,, \\
&\text{(c)}\ \ -\Delta\Omega_1 + \delta\Omega_2 + \bar{\delta}\Omega_5 - D\Omega_6 + \nu\Omega_0 + (\gamma + \bar{\gamma} - 2\mu - \bar{\mu})\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} - 2\tau)\Omega_2 + \\
&\sigma\Omega_3 - \lambda\Omega_4 + \left(-\alpha + \bar{\beta} + 2\pi - \bar{\tau}\right)\Omega_5 + (-\varepsilon - \bar{\varepsilon} + 2\rho + \bar{\rho})\Omega_6 - \kappa\Omega_7 = 0\,,
\end{aligned} \tag{37}$$

with the connections (a): $N_{(4)(2)}$, (b): $N_{(1)(3)}$, and (c): $N_{(2)(1)} + N_{(3)(4)}$.

We project (16) onto null tetrad to construct its NP transcription [18, 23, 27]:

Weyl-Lanczos equations

$$\psi_0 = 2[\delta\Omega_0 - D\Omega_4 + (-\bar{\alpha} - 3\beta + \bar{\pi})\Omega_0 + 3\sigma\Omega_1 + (\bar{\rho} + 3\varepsilon - \bar{\varepsilon})\Omega_4 - 3\kappa\Omega_5],$$

$$2\psi_1 = \Delta\Omega_0 + 3\delta\Omega_1 - \bar{\delta}\Omega_4 - 3D\Omega_5 - (3\gamma + \bar{\gamma} + 3\mu - \bar{\mu})\Omega_0 + 3(-\bar{\alpha} - \beta + \bar{\pi} + \tau)\Omega_1 + 6\sigma\Omega_2 + (3\alpha - \bar{\beta} + 3\pi + \bar{\tau})\Omega_4 + 3(\varepsilon - \bar{\varepsilon} - \rho + \bar{\rho})\Omega_5 - 6\kappa\Omega_6,$$

$$\psi_2 = \Delta\Omega_1 + \delta\Omega_2 - \bar{\delta}\Omega_5 - D\Omega_6 - \nu\Omega_0 - (2\mu - \bar{\mu} + \gamma + \bar{\gamma})\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} + 2\tau)\Omega_2 + \sigma\Omega_3 + \lambda\Omega_4 + (\alpha - \bar{\beta} + 2\pi + \bar{\tau})\Omega_5 - (\varepsilon + \bar{\varepsilon} - \bar{\rho} + 2\rho)\Omega_6 - \kappa\Omega_7, \quad (38)$$

$$2\psi_3 = 3\Delta\Omega_2 + \delta\Omega_3 - 3\bar{\delta}\Omega_6 - D\Omega_7 - 6\nu\Omega_1 + 3(\bar{\mu} - \mu - \bar{\gamma} + \gamma)\Omega_2 + (-\bar{\alpha} + 3\beta + 3\tau + \bar{\pi})\Omega_3 + 6\lambda\Omega_5 + 3(-\alpha - \bar{\beta} + \pi + \bar{\tau})\Omega_6 - (3\varepsilon + \bar{\varepsilon} - \bar{\rho} + 3\rho)\Omega_7,$$

$$\psi_4 = 2[\Delta\Omega_3 - \bar{\delta}\Omega_7 - 3\nu\Omega_2 + (\bar{\mu} + 3\gamma - \bar{\gamma})\Omega_3 + 3\lambda\Omega_6 + (-3\alpha - \bar{\beta} + \bar{\tau})\Omega_7],$$

where were employed the expressions:

$$2\psi_0 = S_{(1)(3)(1)(3)}, \quad 4\psi_1 = S_{(1)(3)(1)(2)} + S_{(1)(3)(4)(3)}, \quad 2\psi_2 = S_{(1)(3)(4)(2)},$$
$$4\psi_3 = S_{(2)(4)(2)(1)} + S_{(2)(4)(3)(4)}, \quad 2\psi_4 = S_{(4)(2)(4)(2)}. \quad (39)$$

If in (38) we use (37) we obtain the set of equations:

$$\psi_0 = 2[\delta\Omega_0 - D\Omega_4 + (-\bar{\alpha} - 3\beta + \bar{\pi})\Omega_0 + 3\sigma\Omega_1 + (\bar{\rho} + 3\varepsilon - \bar{\varepsilon})\Omega_4 - 3\kappa\Omega_5],$$

$$\psi_1 = 2[\delta\Omega_1 - D\Omega_5 - \mu\Omega_0 + (-\bar{\alpha} - \beta + \bar{\pi})\Omega_1 + 2\sigma\Omega_2 + \pi\Omega_4 + (\varepsilon - \bar{\varepsilon} + \bar{\rho})\Omega_5 - 2\kappa\Omega_6],$$

$$\psi_2 = 2[\delta\Omega_2 - D\Omega_6 - 2\mu\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi})\Omega_2 + \sigma\Omega_3 + 2\pi\Omega_5 - (\varepsilon + \bar{\varepsilon} - \bar{\rho})\Omega_6 - \kappa\Omega_7], \quad (40)$$

$$\psi_3 = 2[\Delta\Omega_2 - \bar{\delta}\Omega_6 - 2\nu\Omega_1 + (\bar{\mu} + \gamma - \bar{\gamma})\Omega_2 + \tau\Omega_3 + 2\lambda\Omega_5 + (-\alpha - \bar{\beta} + \bar{\tau})\Omega_6 - \rho\Omega_7],$$

$$\psi_4 = 2[\Delta\Omega_3 - \bar{\delta}\Omega_7 - 3\nu\Omega_2 + (\bar{\mu} + 3\gamma - \bar{\gamma})\Omega_3 + 3\lambda\Omega_6 - (3\alpha + \bar{\beta} - \bar{\tau})\Omega_7].$$

Remark 1: The equations (38) are valid without the condition ${S_{\mu\nu}}^{\alpha}{}_{;\alpha} = 0$. If we ask this differential gauge condition, then we must solve (37) and (38) [or (40)].

Remark 2: If ${S_{\mu\nu}}^{\alpha}{}_{;\alpha} \neq 0$, then (16) is replaced by:

$$S_{\mu\nu\alpha\beta} = S_{\mu\nu\alpha;\beta} - S_{\mu\nu\beta;\alpha} + S_{\alpha\beta\mu;\nu} - S_{\alpha\beta\nu;\mu} + \frac{1}{2}[(H_{\mu\beta} + H_{\beta\mu})g_{\nu\alpha} - (H_{\mu\alpha} + H_{\alpha\mu})g_{\nu\beta} + (H_{\nu\alpha} + H_{\alpha\nu})g_{\mu\beta} - (H_{\nu\beta} + H_{\beta\nu})g_{\mu\alpha}], \quad (41)$$

because now $H_{\mu\nu} \equiv {S_{\mu}}^{\alpha}{}_{\nu;\alpha} \neq H_{\nu\mu}$.

Remark 3: With the equations (40) the system (37) acquires the form:

$$\psi_1 = 2\left[\Delta\Omega_0 - \bar{\delta}\Omega_4 + (\bar{\mu} - 3\gamma - \bar{\gamma})\Omega_0 + 3\tau\Omega_1 + \left(3\alpha - \bar{\beta} + \bar{\tau}\right)\Omega_4 - 3\rho\Omega_5\right],$$

$$\psi_2 = 2\left[\Delta\Omega_1 - \bar{\delta}\Omega_5 - \nu\Omega_0 + (\bar{\mu} - \gamma - \bar{\gamma})\Omega_1 + 2\tau\Omega_2 + \lambda\Omega_4 + \left(\alpha - \bar{\beta} + \bar{\tau}\right)\Omega_5 - 2\rho\Omega_6\right], \quad (42)$$

$$\psi_3 = 2[\delta\Omega_3 - D\Omega_7 - 3\mu\Omega_2 + (-\bar{\alpha} + 3\beta + \bar{\pi})\Omega_3 + 3\pi\Omega_6 - (3\varepsilon + \bar{\varepsilon} - \bar{\rho})\Omega_7].$$

Remark 4: In [18] were obtained the following general solutions for the Weyl-Lanczos equations:

a). Type N in the Petrov classification [8, 20, 21, 31-33].

In this case we can employ a null canonical tetrad [8, 34-36] such that $\psi_j = 0,\ j \neq 4$, $\psi_4 \neq 0$, and the equations (b), (c), (h), (i), (j), (k), (m) and (q) of the set (19), thus:

$$\begin{array}{llll} 2\Omega_0 = \kappa\,, & 2\Omega_3 = -\lambda\,, & 2\Omega_4 = \sigma\,, & 2\Omega_7 = -\nu\,, \\ 6\Omega_1 = \rho\,, & 6\Omega_2 = -\pi\,, & 6\Omega_5 = \tau\,, & 6\Omega_6 = -\mu\,, \end{array} \quad (43)$$

verify (38).

(b). Type III

Now we use a canonical tetrad with the property $\psi_k = 0,\ k \neq 3,\ \psi_3 \neq 0$, then (38) are satisfied by:

$$\begin{array}{llll} \Omega_0 = \kappa\,, & \Omega_3 = -\lambda\,, & \Omega_4 = \sigma\,, & \Omega_7 = -\nu\,, \\ 3\Omega_1 = \rho\,, & 3\Omega_2 = -\pi\,, & 3\Omega_5 = \tau\,, & 3\Omega_6 = -\mu\,. \end{array} \quad (44)$$

We see that (43) and (44) differ by a factor of two.

(c). Type O: $\psi_r = 0,\ r = 0, \dots, 4$, then (43) or (44) are solutions for (38).

It is an open problem to find general solutions of the Weyl-Lanczos relations for the types I, II and D, but we know the Lanczos potential for several spacetimes of interest in general relativity [37-46].

4. CONCLUSIONS

The Newman-Penrose's formalism is very important in Einstein's relativity theory by its applicability in topics as Petrov classification, Lanczos spintensor, embedding of Riemannian spaces, exact solutions, null congruences, Debever-Penrose's principal directions, asymptotic behavior of the gravitational field, etc. This formalism is based in an adequate null tetrad with a natural relationship to 2-spinors.

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