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Constraints and gauge relations in the Lagrangian formalism

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ABSTRACT

For several Lagrangians we show that their local symmetries can be obtained from the associated Euler-Lagrange equations, and we exhibit the explicit presence of the genuine constraints into gauge identities. We also employ the Lanczos approach to Noether's theorem to give connections between the genuine constraints and their time derivatives. Besides, it is evident that the Hamiltonian secondary and tertiary constraints have relationship with the genuine constraints.

***Keywords*:** Singular Lagrangians, Noether's theorem, Gauge identities, Local symmetries, Genuine constraints

1. INTRODUCTION

We consider a physical system where the parameters $q_1, q_2, \ldots, q_n$ are its generalized coordinates, that is, there are n degrees of freedom. The action:

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt, \qquad \dot{q} = \frac{dq}{dt}, \tag{1}$$

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is fundamental in the dynamical evolution of the system. We can change to new coordinates via the local transformations:

$$\tilde{t} = t, \qquad \tilde{q}_i = q_i + \varepsilon\, \alpha_i(q,t), \qquad i = 1, 2, \dots, n \tag{2}$$

where: ε is an infinitesimal parameter, thus the action takes the value:

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}\,, \frac{d\tilde{q}}{d\tilde{t}})\, d\tilde{t}. \tag{3}$$

If $\delta S = \delta\tilde{S}$, to first order in ε, then we say that the action is invariant under the transformation (2), that is, (2) are local symmetries associated with the corresponding Lagrangian. The variation of the action (1) is [1]:

$$\delta S = -\int_{t_1}^{t_2} dt\, E_i\, \delta q_i = -\int_{t_1}^{t_2} dt\, \vec{E} \cdot \delta\vec{q}, \quad \delta q_i\big(t_j\big) = 0, \;\; j = 1, 2 \tag{4}$$

where: the Dedekind (1868)-Einstein [2, 3] summation convention is used for repeated indices, with the Euler derivatives:

$$E_i(q, \dot{q}, \ddot{q}) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, \dots, n \tag{5}$$

therefore $\delta S = 0$ if :

$$\delta\vec{q} \cdot \vec{E} = -\frac{d(\varepsilon Q)}{dt}\,, \qquad Q(t_2) = Q(t_1) = 0, \tag{6}$$

without the participation of the equations of motion $E_i = 0$.

In Sec. 2 exhibit the gauge identities [4] (6) for four Lagrangians studied in [5] to show that the quantity Q is in terms of the corresponding genuine constraints, which depend of the Hamiltonian secondary and tertiary constraints [4, 6]. For these degenerate Lagrangians the transformation (2) have arbitrary functions, that is, they are gauge symmetries [4, 6, 7], then we apply the Lanczos approach [1, 5, 8-12] to Noether's theorem [13-18] to deduce that the 'conservation laws' associated to these arbitrary functions are relations between the genuine constraints and their time derivatives. We know that the gauge transformations no need the equations of motion, and that they can be obtained via, for example, a matrix method [4, 5, 19] or as solutions of the Killing equations [5, 20, 21], however, in Sec. 3 we indicate a simple alternative deduction based in the Euler-Lagrange equations.

2. GENUINE CONSTRAINTS AND GAUGE IDENTITIES

In [5] the matrix method [4, 19] was applied to several singular Lagrangians to construct their gauge identities which imply the corresponding gauge symmetries (2) in according with (6). Besides, this method gives the genuine constraints on shell, that is, on the subspace of physical trajectories where are valid the equations of motion. For each Lagrangian we only exhibit the essential results because our principal aim is to show that the function Q at (6) depends of the genuine constraints, and that the 'conserved quantities' for these gauge transformations have relationship with the genuine constraints and their derivatives.

a). Rothe [4]:

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \tag{7}$$

here we have [5] one gauge identity:

$$E_1 + E_2 + \frac{d}{dt}E_2 = 0, \tag{8}$$

and one genuine constraint:

$$\varphi = E_2 = -q_2 - \dot{q}_1 + q_1, \tag{9}$$

which only depends on the coordinates and velocities, and vanish on shell. We multiply (8) by $\varepsilon\, \alpha_1(t)$ to obtain:

$$\varepsilon\, \alpha_1\, E_1 + \varepsilon\, (\alpha_1 - \dot{\alpha}_1)E_2 = -\frac{d}{dt}(\varepsilon\, \alpha_1 \varphi), \tag{10}$$

whose comparison with (6) gives $\delta q_1 = \varepsilon\, \alpha_1$ and $\delta q_2 = \varepsilon\, (\alpha_1 - \dot{\alpha}_1)$, thus (2) implies the local transformation:

$$\tilde{t} = t\,, \quad \tilde{q}_1 = q_1 + \varepsilon\, \alpha_1\,, \quad \tilde{q}_2 = q_2 + \varepsilon\, (\alpha_1 - \dot{\alpha}_1), \quad Q = \alpha_1\, \varphi\,, \tag{11}$$

where: α_1 is an arbitrary function; the number of independent arbitrary functions present in the point symmetry equals the number of gauge identities.

Noether [7, 13-18, 22-24] proved that in a variational principle the existence of symmetries implies the presence of conservation laws [25, 26]. Here we shall apply this theorem, but in the Lanczos version [1, 5, 8-12]: First, in the transformation (11) we accept that $\alpha_1 = a =$ constant, thus we have the global symmetry $\tilde{t} = t,\ \tilde{q}_1 = q_1 + \varepsilon\, a,\ \tilde{q}_2 = q_2 + \varepsilon\, a$. Now we change the parameter a by the function $\beta(t)$, our new degree of freedom, to obtain the local symmetry $\tilde{t} = t,\ \tilde{q}_1 = q_1 + \varepsilon\, \beta(t),\ \tilde{q}_2 = q_2 + \varepsilon\, \beta(t)$, therefore $\tilde{L} = L + \varepsilon\left[\beta\, \dot{q}_1 + \dot{\beta}\, (\dot{q}_1 + q_2)\right]$, and from the Euler-Lagrange equation for β, $\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\beta}}\right) - \frac{\partial \tilde{L}}{\partial \beta} = 0$, we deduce that:

$$\frac{d}{dt}(-\dot{q}_1 - q_2 + q_1) = \frac{d}{dt}\varphi = 0, \tag{12}$$

which is correct for the genuine constraint on the subspace of physical paths ($\vec{E} = \vec{0}$); the expression (12) is a conserved quantity associated with (11).

The Dirac's method [4, 6, 7, 27-34] gives the Hamiltonian constraints:

$$\phi_1 = p_2 \quad \text{(primary)}, \qquad \phi_2 = p_1 - q_1 \quad \text{(secondary)}, \tag{13}$$

with the canonical momentum $p_1 = \dot{q}_1 + q_2$, then from (9):

$$\varphi = -\phi_2 \,, \tag{14}$$

which represents a relationship between the genuine and secondary constraints because both depend of the equations of motion.

b). Henneaux-Teitelboim [4, 35]:

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 \,, \tag{15}$$

in this case we find [5] one gauge identity:

$$\dot{q}_1^2 E_1 + \dot{q}_1 e^{q_1} E_2 + e^{q_1} E_3 - \dot{q}_1 \frac{d}{dt} E_1 + \frac{d}{dt}\left(-\dot{q}_1 E_1 - e^{q_1} E_2 + \frac{d}{dt} E_1\right) = 0, \tag{16}$$

and two genuine constraints:

$$\varphi_1 = e^{q_1}(\dot{q}_2 - e^{q_1}), \qquad \varphi_2 = e^{q_1}(q_2 - \dot{q}_3) \,. \tag{17}$$

We multiply (16) by $\varepsilon\, \alpha_3(t) e^{-q_1}$ to deduce the expression:

$$\varepsilon\, \ddot{\alpha}_3 e^{-q_1} E_1 + \varepsilon\, \dot{\alpha}_3 E_2 + \varepsilon\, \alpha_3 E_3 = -\frac{d}{dt}[\varepsilon e^{-q_1}(\alpha_3 \varphi_2 - \dot{\alpha}_3 \varphi_1)], \tag{18}$$

and its comparison with (6) permits to obtain that $\delta q_1 = \varepsilon\, \ddot{\alpha}_3 e^{-q_1}$, $\delta q_2 = \varepsilon\, \dot{\alpha}_3$ and $\delta q_3 = \varepsilon\, \alpha_3$, that is, the gauge symmetry:

$$\tilde{t} = t, \quad \tilde{q}_1 = q_1 + \varepsilon\, e^{-q_1} \ddot{\alpha}_3 \,, \quad \tilde{q}_2 = q_2 + \varepsilon\, \dot{\alpha}_3 \,, \quad \tilde{q}_3 = q_3 + \varepsilon\, \alpha_3 \,,$$
$$Q = e^{-q_1}(\alpha_3 \varphi_2 - \dot{\alpha}_3 \varphi_1), \tag{19}$$

being α_3 an arbitrary function.

In according with the Lanczos approach to Noether's theorem, into the transformation (19) we utilize $\alpha_3 = a =$ constant, to deduce the global symmetry $\tilde{t} = t$, $\tilde{q}_1 = q_1$, $\tilde{q}_2 = q_2$, $\tilde{q}_3 = q_3 + \varepsilon\, a$. After in (15) we apply this symmetry when $a \to \beta(t)$:

$$\tilde{L} = L + \varepsilon\, \dot{\beta}\, (\dot{q}_3 - q_2) \quad \therefore \quad \frac{d}{dt}(e^{-q_1} \varphi_2) = 0, \;\; \text{that is,} \;\; \frac{d}{dt}\varphi_2 - \dot{q}_1 \varphi_2 = 0, \tag{20}$$

which is a conservation law associated with (19), on shell.

For (15) the Hamiltonian formalism implies three constraints:

$$\phi_1 = p_1 \;\; \text{(primary)}, \quad \phi_2 = p_2 \;\; \text{(secondary)}, \quad \phi_3 = p_3 \;\; \text{(tertiary)}, \tag{21}$$

with the canonical momenta $p_2 = \dot{q}_2 - e^{q_1}$ and $p_3 = \dot{q}_3 - q_2$, thus from (17):

$$\varphi_1 = e^{q_1}\phi_2\,, \qquad\qquad \varphi_2 = -e^{q_1}\phi_3\,, \tag{22}$$

then the genuine constraints are in terms of the secondary and tertiary Hamiltonian constraints.

c). Havelková [20] – Torres del Castillo [36]:

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3\,, \tag{23}$$

with one gauge identity [5]:

$$E_2 + \frac{d}{dt}\left(E_1 - \frac{d}{dt}E_2\right) = 0, \tag{24}$$

and two genuine constraints:

$$\varphi_1 = \dot{q}_3, \qquad \varphi_2 = -q_3. \tag{25}$$

We multiply (24) by $\varepsilon\,\alpha$ to obtain the relation:

$$\varepsilon\,\dot{\alpha}E_1 + \varepsilon\,(\ddot{\alpha} - \alpha)\,E_2 = \varepsilon\,\frac{d}{dt}[\dot{\alpha}\,\varphi_1 + \alpha\,\varphi_2], \tag{26}$$

then from (6) results that $\delta q_1 = \varepsilon\,\dot{\alpha}$, $\delta q_2 = \varepsilon\,(\ddot{\alpha} - \alpha)$ and $\delta q_3 = 0$, which means the gauge symmetry:

$$\tilde{t} = t, \quad \tilde{q}_1 = q_1 + \varepsilon\,\dot{\alpha}, \quad \tilde{q}_2 = q_2 + \varepsilon\,(\ddot{\alpha} - \alpha), \quad \tilde{q}_3 = q_3, \quad Q = -(\dot{\alpha}\varphi_1 + \alpha\varphi_2), \tag{27}$$

with α an arbitrary function. The Lanczos method [1, 5, 8-12] to find conservation laws is applied to the point symmetry (27), which implies:

$$\varphi_1 = 0 \qquad \text{(on shell)}. \tag{28}$$

For (23) the Dirac's technique [4, 6, 7, 27-34] gives three constraints:

$$\phi_1 = p_2 \quad \text{(primary)}, \quad \phi_2 = p_1 \quad \text{(secondary)}, \quad \phi_3 = q_3 \quad \text{(tertiary)}, \tag{29}$$

with the canonical momentum $p_1 = \dot{q}_3$, then from (25):

$$\varphi_1 = \phi_2\,, \qquad \varphi_2 = -\phi_3\,, \tag{30}$$

thus we see again the relationship between genuine and Hamiltonian constraints.

d). Rothe [4]:

$$L = \frac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\,\dot{q}_1 + \frac{1}{2}\,(q_1 - q_2 + q_3)^2, \tag{31}$$

with two gauge identities [5]:

$$E_2 + E_3 = 0, \qquad E_1 + E_2 + \frac{d}{dt} E_2 = 0, \tag{32}$$

and one genuine constraint:

$$\varphi = -\dot{q}_1 + q_1 - q_2 + q_3. \tag{33}$$

Each linearly independent gauge identity is multiplied by an arbitrary function and the sum of these expressions permits to obtain (6):

$$\varepsilon\, \alpha_1\, E_1 + \varepsilon\, (\alpha_3 - \dot{\alpha}_1 + \alpha_1)\, E_2 + \varepsilon\, \alpha_3\, E_3 = -\frac{d}{dt}(\varepsilon\, \alpha_1\, \varphi), \tag{34}$$

which leads to the point transformation:

$$\tilde{t} = t, \quad \tilde{q}_1 = q_1 + \varepsilon\, \alpha_1\,, \quad \tilde{q}_2 = q_2 + \varepsilon\, (\alpha_1 - \dot{\alpha}_1 + \alpha_3), \quad \tilde{q}_3 = q_3 + \varepsilon\, \alpha_3\,, \\ Q = \alpha_1\, \varphi, \tag{35}$$

being α_1 and α_3 arbitrary functions. In this case, the Lanczos technique gives the conservation law:

$$\frac{d}{dt}\varphi = 0, \qquad \left(\vec{E}^{(0)} = \vec{0}\right), \tag{36}$$

that is, the constraints must be consistent with the time evolution.

For this Lagrangian (31) the Hamiltonian formalism gives three constraints:

$$\phi_1 = p_2 \quad \text{(primary)}, \quad \phi_2 = p_3 \;\; \text{(primary)}, \quad \phi_3 = p_1 - q_1 \quad \text{(secondary)}, \tag{37}$$

with the canonical momentum $p_1 = \dot{q}_1 + q_2 - q_3$, thus from (33):

$$\varphi = -\phi_3\,. \tag{38}$$

The relations (11), (19), (27) and (35) indicate that the quantity Q in (6) can be written in terms of the genuine constraints. The expressions (12), (20), (28) and (36) exhibit that the ‘conservation laws’ associated with the arbitrary functions of the gauge symmetries, determined via the Lanczos approach to Noether’s theorem, have relationship with the genuine constraints and their time derivatives. From (14), (22), (30) and (38) we see the connection of the genuine constraints with the secondary and tertiary Hamiltonian constraints.

Here we used gauge transformations obtained [5] with the matrix method [4, 5, 19] without to utilize the Euler-Lagrange equations. In Sec. 3 we exhibit a simple alternative deduction of these gauge symmetries, where it is necessary to employ the equations of motion.

3. AN ALTERNATIVE APPROACH TO POINT SYMMETRIES

We shall show that for each Lagrangian studied in Sec. 2 their equations of motion permit to deduce the corresponding local transformations.

a). Lagrangian (7): In this case the Euler-Lagrange equations imply the relations

$$\ddot{q}_1 - q_1 + q_2 + \dot{q}_2 = 0 \quad \& \quad q_1 - q_2 - \dot{q}_1 = 0 \quad \therefore \quad \dot{q}_2 = \dot{q}_1 - \ddot{q}_1 \,, \tag{39}$$

in connection with (2) we make the identification:

$$\dot{q}_r = \frac{1}{\varepsilon}(\tilde{q}_r - q_r), \qquad r = 1, 2 \tag{40}$$

but rank (Hessian matrix) = rank ($\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}$) = 1, then in the equations of motion we have one arbitrary function, hence we take $\dot{q}_1 = \alpha_1$ and (39) gives $\dot{q}_2 = \alpha_1 - \dot{\alpha}_1$, which generates the transformation (11).

b). Lagrangian (15): From the variational equations we obtain the expressions

$$\dot{q}_1 = e^{-q_1}\ddot{q}_2 \,, \qquad \dot{q}_2 = \ddot{q}_3 \,, \tag{41}$$

with one arbitrary function because rank (Hessian matrix) = 2 for three degrees of freedom, then we select

$\dot{q}_3 = \alpha_3$ and (41) implies $\dot{q}_1 = e^{-q_1}\ddot{\alpha}_3$ & $\dot{q}_2 = \dot{\alpha}_3$, in harmony with the gauge symmetry (19).

c). Lagrangian (23): The dynamical equations permit to write the relations

$$\dot{q}_2 = \ddot{q}_1 - q_1 \,, \qquad \dot{q}_3 = 0 \tag{42}$$

but rank (Hessian matrix) = 2, which gives one arbitrary function, hence we take $\dot{q}_1 = \dot{\alpha}$ and from (42) we deduce that $\dot{q}_2 = \ddot{\alpha} - \alpha$, in according with the local transformation (27).

d). Lagrangian (31): The Euler-Lagrange equations imply the expression

$$\dot{q}_2 = \dot{q}_1 - \ddot{q}_1 + \dot{q}_3 \,, \tag{43}$$

such that rank (Hessian matrix) = 1 for three degrees of freedom, therefore there are two arbitrary functions, hence we select $\dot{q}_1 = \alpha_1$, $\dot{q}_3 = \alpha_3$ and from (43) $\dot{q}_2 = \alpha_1 - \dot{\alpha}_1 + \alpha_3$ which is equivalent to the gauge symmetry (35).

4. CONCLUSION

Thus we observe that, for our four Lagrangians, the equations of motion and the identification (40) permit to find their local transformations (2), which is a simple alternative deduction to matrix method [4, 5, 19].

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