On some triple sequence spaces of $\chi^3$

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ABSTRACT

We introduce the concepts in probability of rough lacunary statistical convergence and $N_0$ - rough convergence of a triple sequence spaces of real numbers and discuss general properties of above rough convergence.

Keywords: Rough lacunary statistical convergence, triple sequences, $N_0$ - convergence, chi sequence

1. INTRODUCTION

The idea of rough convergence was introduced by Phu \cite{11}, who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar \cite{1} extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. \cite{10} extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated.
at the initial by Sahiner et al. [12,13], Esi et al. [2-4], Datta et al. [5], Subramanian et al. [14], Debnath et al. [6], Esi et al. [16] and many others.

A triple sequence \( x = (x_{m,n,k}) \) is said to be triple analytic if

\[
\sup_{m,n,k} |x_{m,n,k}|^{\frac{1}{m+n+k}} < \infty.
\]

The space of all triple analytic sequences are usually denoted by \( \Lambda^3 \). A triple sequence \( x = (x_{m,n,k}) \) is called triple gai sequence if

\[
((m + n + k)! |x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.
\]

The space of all triple gai sequences are usually denoted by \( \chi^3 \).

2. DEFINITIONS AND PRELIMINARIES
2.1. Definition

An Orlicz function (see [7]) is a function \( M: [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \), for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \), then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence \( g = (g_{mn}) \) defined by

\[
g_{mn}(v) = \sup \{ |v| - (f_{mn})(u) : u \geq 0 \}, m,n,k = 1,2, \ldots
\]

is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak-Orlicz function \( f \), [see [9] ] the Musielak-Orlicz sequence space \( t_f \) is defined as follows

\[
t_f = \{ x \in w^3 : l_f(|x_{m,n,k}|)^{1/(m+n+k)} \to 0 \text{ as } m,n,k \to \infty \},
\]

where: \( l_f \) is a convex modular defined by

\[
l_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mn,k}(|x_{m,n,k}|)^{1/(m+n+k)}, x = (x_{m,n,k}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric

\[
d(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{f_{mn,k}(|x_{m,n,k}|^{1/(m+n+k)})}{mnk}
\]

is an extended real number.
2. Definition

Let $X, Y$ be a real vector space of dimension $w$, where $n \leq m$. A real valued function $d_p(x_1, ..., x_n) = \| (d_1(x_1, 0), ..., d_n(x_n, 0)) \|_p$ on $X$ satisfying the following four conditions:

(i) $\| (d_1(x_1, 0), ..., d_n(x_n, 0)) \|_p = 0$ if and only if $d_1(x_1, 0), ..., d_n(x_n, 0)$ are linearly dependent,

(ii) $\| (d_1(x_1, 0), ..., d_n(x_n, 0)) \|_p$ is invariant under permutation,

(iii) $\| (\alpha d_1(x_1, 0), ..., d_n(x_n, 0)) \|_p = |\alpha| \| (d_1(x_1, 0), ..., d_n(x_n, 0)) \|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) = (d_x(x_1, x_2, ..., x_n)^p + d_y(y_1, y_2, ..., y_n)^p)^{1/p} \text{ for } 1 \leq p < \infty; \text{ (or)}$

(v) $d((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) = \sup \{d_x(x_1, x_2, ..., x_n), d_y(y_1, y_2, ..., y_n)\}$,

for $x_1, x_2, ..., x_n \in X, y_1, y_2, ..., y_n \in Y$ is called the $p$ product metric of the Cartesian product of $n$ metric spaces (see [15])

2.3. Definition

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

\[
m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty \quad \text{and} \quad n_0 = 0, h_\ell = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty. \quad k_0 = 0, h_j = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.
\]

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i h_\ell h_j$, and $\theta_{i,\ell,j}$ is determined by

\[
l_{i,\ell,j} = \{(m, n, k): m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\}, q_i = \frac{m_i}{m_{i-1}}, q_\ell = \frac{n_\ell}{n_{\ell-1}}, q_j = \frac{k_j}{k_{j-1}}.
\]

2.4. Definition

Let $\theta = \{m_r n_s k_t\}_{(r,s,t) \in \mathbb{N}^3}$ be the triple lacunary sequence spaces. A triple sequence spaces of $(X_{mnk})$ is said to be triple lacunary statistically convergent to a number $X$ if for any $\varepsilon > 0$,

\[
\lim_{r,s,t \to \infty} \frac{1}{h_{r,s,t}} \left| \{(m, n, k) \in l_{r,s,t}: |X_{mnk} - X| \geq \varepsilon\} \right| = 0,
\]

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where: \( l_{r,s,t} = \{(m, n, k): m_{r-1} < m < m_r \quad \text{and} \quad n_{s-1} < n \leq n_s \quad \text{and} \quad k_{t-1} < k \leq k_t\} \), \( q_r = \frac{m_r}{m_{r-1}}, q_s = \frac{n_s}{n_{s-1}}, q_t = \frac{k_t}{k_{t-1}} \).

In this case write
\[
\|X^3_f(X_{mnk}), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \rightarrow S^\theta \|X^3_f(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p.
\]

2. 5. Definition

Let \( \theta = \{m_r n_s k_t\} \subseteq \mathbb{N}^u \) be the triple sequence spaces of lacunary. A number
\( (X_{mnk}) \) is said to be \( N_{\theta} \)-convergent to a real number \( X \) if for every \( \varepsilon > 0 \),
\[
\lim_{rst \to \infty} \frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} |X_{mnk}, 0| = 0.
\]
In this case we write \( X_{mnk} \to N_{\theta} 0 \).

2. 6. Definition

Let \( \alpha \) be a nonnegative real number. A triple sequence spaces of random variables
\( (X_{mnk}) \) is said to be rough lacunary statistically convergent in probability to \( X: W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) with respect to the roughness of degree \( \alpha \) if for any \( \varepsilon, \delta > 0 \),
\[
\lim_{rst \to \infty} \frac{1}{h_{rst}} \left( \frac{(m, n, k) \in I_{rst}}{P \left( \left\{ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right\} \geq \alpha + \varepsilon \right) \right) = 0
\]
and we write \( \|X^3_f(X_{mnk}), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to S^\alpha \alpha \). It will be denoted by \( \alpha S^\alpha \).

2. 7. Definition

Let \( \alpha \) be a nonnegative real number. A triple sequence variables of random variables
\( (X_{mnk}) \) is said to be rough \( N_{\theta} \)-convergent in probability to \( X: W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) with respect to the roughness of degree
\[
\alpha \quad \text{if for any} \quad \varepsilon > 0, \quad \lim_{rst \to \infty} \frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} P(\| f_{mnk}(\|\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \geq \alpha + \varepsilon) = 0,
\]
and we write
\[
\|X^3_f(X_{mnk}), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to N^{\alpha \theta}_{\alpha} \|X^3_f(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p.
\]

The class of all \( \alpha - N_{\theta} \)-convergent triple sequence spaces of random variables in probability will be denoted by \( \alpha N^{\alpha \theta}_{\alpha} \).

2. 8. Definition

Let \( \theta = \{m_r n_s k_t\} \subseteq \mathbb{N}^u \) be lacunary triple sequence spaces of lacunary refinement of
\( \theta \) is a triple lacunary sequence spaces of \( \theta' = \{m'_r n'_s k'_t\} \subseteq \mathbb{N}^u \) satisfying
\[
\theta = \{m_r n_s k_t\} \subseteq \{m'_r n'_s k'_t\} \subseteq \{m''_r n''_s k''_t\} \subseteq \{m'''_r n'''_s k'''_t\} \subseteq \mathbb{N}^u.
\]
2. 9. Note

Let $f$ be an Musielak-Orlicz function and triple sequence spaces of $\| \chi_f^3, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p = [f_{mnk}(\| \mu_{mnk}(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p)]$, where $\mu_{mnk}(X) = (((m + n + k)! X_{mnk})^{1/(m+n+k)}, 0)$.

3. MAIN RESULTS

3. 1. Theorem

Let $\theta = \{m_r, n_s, k_t\}_{(rst) \in \mathbb{N}^3}$ be a triple lacunary sequence spaces. Then the followings are equivalent:

(i) $\| \chi_f^3(X_{mnk}), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p$ is $\alpha$ - triple lacunary statistically convergent in probability to $0$.

(ii) $\| \chi_f^3(X_{mnk}), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p$ is $\alpha - N_\theta$ convergent in probability to $0$.

Proof: (i) $\Rightarrow$ (ii) First suppose that $\| \chi_f^3(X_{mnk}), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to_\alpha^p 0$. Then we can write

$$\frac{1}{h_{rst} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t}} \left\{ P\left( \| f_{mnk}(\| \mu_{mnk}(X) - \mu(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p) \geq \alpha + \epsilon \right) \right\} = \frac{1}{h_{rst} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t}} \left\{ P\left( \| f_{mnk}(\| \mu_{mnk}(X) - \mu(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p) \geq \alpha + \epsilon \right) \right\} + \frac{1}{h_{rst} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t}} \left\{ P\left( \| f_{mnk}(\| \mu_{mnk}(X) - \mu(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p) \geq \alpha + \epsilon \right) \right\} \leq \frac{1}{h_{rst}} \left\{ P\left( \| f_{mnk}(\| \mu_{mnk}(X) - \mu(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p) \geq \alpha + \epsilon \right) \right\} \geq \frac{\delta}{2}.$$

(ii) $\Rightarrow$ (i) Next suppose that condition (ii) holds. Then

$$\sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} \left\{ P\left( \| f_{mnk}(\| \mu_{mnk}(X) - \mu(X), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p) \geq \alpha + \epsilon \right) \right\} \geq \frac{\delta}{2}.$$
Therefore
\[
\frac{1}{\delta} h_{rst} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \frac{1}{h_{rst}} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]

Hence \( \| \chi^3_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0} \).

### 3.2. Theorem

If \( \| \chi^3_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0} \) and
\[
\| \chi^3_{Y_{mnk}}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0} \text{ then}
\]
P\( \{ \| \chi^3_{mnk} - Y_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \geq \alpha + \varepsilon \} \) = \bar{0}.

**Proof:** Omitted.

### 3.3. Theorem

Let \( \theta' = \{ m_r n_s k_t \}_{(rst) \in N \cup 0} \) be a triple lacunary refinement of the triple lacunary sequence spaces of \( \theta = \{ m_r n_s k_t \}_{(rst) \in N \cup 0} \). Let \( h_r = (m_{r-1}, m_r], h_s = (n_{s-1}, n_s], h_t = (k_{t-1}, k_t], r, s, t = 1,2,3 \ldots \). If there exists a \( \eta > 0 \) such that \( \frac{|h_{rst}|}{|l_{rst}|} > \eta \) for every \( h_{rst} \subseteq l_{rst} \).

Then
\[
\| \chi^3_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0} \Rightarrow
\]
\[
\| \chi^3_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0}.
\]

**Proof:** Let \( \| \chi^3_{mnk}, (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \to \frac{S^p}{s} \bar{0} \) and \( \varepsilon, \delta > 0 \). Therefore
\[
\lim_{rst \to \infty} \frac{1}{|l_{rst}|} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]
For every \( h_{rst} \) we can find \( l_{rst} \) such that \( h_{rst} \subseteq l_{rst} \). We obtain
\[
\frac{1}{|l_{rst}|} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]
\[
\frac{1}{|l_{rst}|} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]
\[
\frac{1}{|l_{rst}|} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]
\[
\frac{1}{|l_{rst}|} \left\{ P\{ \| f_{mnk}(\mu_{mnk}(X) - \mu(X)), (d(x_1), d(x_2), \ldots, d(x_{n-1})) \|_p \} \geq \alpha + \varepsilon \} \right\} \geq \delta \}
\]
3.4. Remark

In this refinements $\theta'$ of $\theta$ exists. The following example, let $(u, v, w) \in \mathbb{N}\backslash\{1,1,1\}$ and introducing $(u - 1, v - 1, w - 1)$ points in the interval $h_r = (m_{r-1}, m_r], h_s = (n_{s-1}, n_s], h_t = (k_{t-1}, k_t], r, s, t = 1, 2, 3 \ldots$.

Then

$$h_1 = \left( (m_0, n_0, k_0), (m_0, n_0, k_0) + \frac{j_1}{uvw} \right) h_2$$

$$= \left( (m_0, n_0, k_0) + \frac{j_1}{uvw}, (m_0, n_0, k_0) + \frac{2j_1}{uvw} \right) 0.5cm : h_{uvw}$$

$$= \left( (m_0, n_0, k_0) + \frac{(u-1)(v-1)(w-1)j_1}{uvw}, (m, n, k) \right) h_{(u+1,v+1,w+1)}$$

$$= \left( (m_1, n_1, k_1), (m_1, n_1, k_1) + \frac{j_1}{uvw} \right) h_{(u+2,v+2,w+2)}$$

$$= \left( (m_1, n_1, k_1) + \frac{j_2}{uvw}, (m, n, k) + \frac{2j_2}{uvw} \right) 0.5cm : h_{(2u,2v,2w)}$$

$$= \left( (m_1, n_1, k_1) + \frac{(u-1)(v-1)(w-1)j_2}{uvw}, (m_2, n_2, k_2) \right) 0.5cm$$

$\vdash h_{r-1(u+1,v+1,w+1)}$

$$= \left( (m_{r-1}, n_{r-1}, k_{r-1}), (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{h_r}{uvw} \right) h_{r-1(u+2,v+2,w+2)}$$

$$= \left( (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{h_r}{uvw}, (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{2h_r}{uvw} \right) 0.5cm$$

$\vdash h_{r-1(2u,2v,2w)}$

$$= \left( (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{(u-1)(v-1)(w-1)h_r}{uvw}, (m_r, n_r, k_r) \right) . 0.5cm$$

Then $|h_{rst}| \to \infty$ as $r, s, t \to \infty$ and $\frac{|h_{rst}|}{|l_{rst}|} \geq \frac{1}{uvw}$ for every $h_{rst} \subseteq l_{rst}$.

4. CONCLUSIONS

We introduced triple sequence spaces of rough lacunary statistical convergence in probability and triple sequence spaces of $N_0$ — rough convergence with respect sequence of Musielak-Orlicz function. For the reference sections, consider the following introduction described the main results are motivating the research.
References


