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# Spinor representation of the electromagnetic field

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**ABSTRACT**

We exhibit the tensor, spinor and Newman-Penrose (NP) forms of Faraday tensor and the Maxwell equations. Our approach is based in real and null tetrads, which allows introduce in natural manner the NP and 2-spinor techniques to study arbitrary electromagnetic fields, with special emphasis in the relationship between quaternions and Lorentz transformations.

***Keywords*:** Newman-Penrose formalism, Faraday's tensor, Maxwell equations, Maxwell spinor, Quaternions, Lorentz transformations

## 1. INTRODUCTION

In Minkowski spacetime, we have the metric tensor $\left(g_{\mu\nu}\right) = Diag\,(1, -1, -1, -1)$ with the quaternion of position:

$$\mathbf{R} = \frac{1}{\sqrt{2}}(c\,t - i\,x\,\mathbf{I} - i\,y\,\mathbf{J} - i\,z\,\mathbf{K}), \qquad i = \sqrt{-1}\,, \tag{1}$$

and the corresponding Lorentz transformations are generated by means of the Klein-Sommerfeld's expression [1-6]:

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$$\widetilde{\mathbf{R}} = \mathbf{A}\,\mathbf{R}\,\overline{\mathbf{A}}^{*}\,, \tag{2}$$

where $\mathbf{A} = a_0 + a_1\,\mathbf{I} + a_2\,\mathbf{J} + a_3\,\mathbf{K}$ is an arbitrary unit quaternion, and:

$$\overline{\mathbf{A}}^{*} = a_0^{*} - a_1^{*}\,\mathbf{I} - a_2^{*}\,\mathbf{J} - a_3^{*}\,\mathbf{K}\,. \tag{3}$$

The formula (2) was obtained by Hamilton [7] and Cayley [8] for 3-rotations [9, 10], in such a case the quantities $a_\mu,\ \mu = 0, \dots, 3$ are real and do match with the Euler-Olinde Rodrigues parameters [11-13].

The matrix version of (2) will be the starting point of our spinorial analysis, and it can be deduced by means of the isomorphism introduced by Cayley [14, 15] between the quaternion basis elements and the Cayley [14]-Sylvester [16]-Pauli [17] matrices:

$$1 \leftrightarrow \mathrm{I}_{2x2}\,, \qquad \mathbf{I} \leftrightarrow i\,\sigma_1\,, \qquad \mathbf{J} \leftrightarrow -i\,\sigma_2\,, \qquad \mathbf{K} \leftrightarrow i\,\sigma_3\,, \tag{4}$$

so that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Thus, the quaternion **A** is isomorphic to the complex matrix 2x2 [11, 18]:

$$\mathrm{A} = \begin{pmatrix} a_0 + ia_3 & -a_2 + ia_1 \\ a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0\,\mathrm{I} + i\,a_1\sigma_1 - i\,a_2\sigma_2 + i\,a_3\sigma_3\,, \tag{6}$$

moreover

$$\mathbf{A}\,\overline{\mathbf{A}} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \qquad \Longrightarrow \qquad det\ \mathrm{A} = 1\,, \tag{7}$$

as is the case when (2) is employed to build Lorentz transformations. With the aid of (6), it is straightforward to find the matrices associated with the quaternions (1) and (3):

$$\mathbf{R} \leftrightarrow \mathrm{X} = \frac{1}{\sqrt{2}}\begin{pmatrix} ct + z & x + iy \\ x - iy & ct - z \end{pmatrix}, \quad \overline{\mathbf{A}}^{*} \leftrightarrow \mathrm{A}^{\dagger} = \mathrm{A}^{\mathrm{T}^{*}} = \begin{pmatrix} a_0^{*} - ia_3^{*} & a_2^{*} - ia_1^{*} \\ -a_2^{*} - ia_1^{*} & a_0^{*} + ia_3^{*} \end{pmatrix}, \tag{8}$$

then (2) leads to the Cartan's expression [19, 20]:

$$\widetilde{\mathrm{X}} = \mathrm{A}\,\mathrm{X}\,\mathrm{A}^{\dagger}\,, \qquad det\ \mathrm{A} = det\,\mathrm{A}^{\dagger} = 1\,, \tag{9}$$

also obtained by Olinde Rodrigues [21] for 3-rotations.

Taking the determinant of (9) we deduce the conservation of Minkowski's interval:

$$c^2\tilde{t}^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = c^2t^2 - x^2 - y^2 - z^2\,, \tag{10}$$

which implies [12] the linearity of the coordinate transformation between both reference frames (we use the summation convention on repeated indices introduced by Dedekind (1868) [23, 24] and Einstein):

$$\tilde{x}^{\mu} = \mathrm{L}^{\mu}{}_{\nu}\, x^{\nu}, \qquad (x^{\mu}) = (ct, x, y, z), \tag{11}$$

where $\mathrm{L} = (\mathrm{L}^{\alpha}{}_{\beta})$ is an element of the homogeneous Lorentz group with $det\,\mathrm{L} = 1$. The substitution of (11) into (9) provides explicit formulas for $\mathrm{L}^{\mu}{}_{\nu}$ in terms of the Euler-Olinde Rodrigues parameters, see [4, 22, 25, 26]. The matrices ±A lead to the same L, therefore they constitute a bi-representation of the Lorentz transformations.

In (9) we have the 2-spinor (Ehrenfest introduced the term spinor, see [27-29]):

$$\left(\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}\right) = \begin{pmatrix} \mathrm{X}^{1\dot{1}} & \mathrm{X}^{1\dot{2}} \\ \mathrm{X}^{2\dot{1}} & \mathrm{X}^{2\dot{2}} \end{pmatrix}, \qquad \mathrm{X}^{1\dot{1}} = \frac{1}{\sqrt{2}}(x^0 + x^3), \qquad \mathrm{X}^{1\dot{2}} = \frac{1}{\sqrt{2}}(x^1 + i\,x^2),$$
$$\mathrm{X}^{2\dot{1}} = \frac{1}{\sqrt{2}}(x^1 - i\,x^2), \qquad \mathrm{X}^{2\dot{2}} = \frac{1}{\sqrt{2}}(x^0 - x^3), \qquad \overline{\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}} = \mathrm{X}^{\mathrm{B}\dot{\mathrm{A}}}, \tag{12}$$

it is evident the Hermitian character of X; furthermore, one of its indices transforms (under a Lorentz mapping) according to A, meanwhile the dotted index does so via $\mathrm{A}^{\dagger}$:

$$\widetilde{\mathrm{X}^{\mathrm{B}\dot{\mathrm{C}}}} = \mathrm{A}^{\mathrm{B}}{}_{\mathrm{D}}\, \mathrm{X}^{\mathrm{D}\dot{\mathrm{E}}}\, \mathrm{A}^{\dagger}{}_{\dot{\mathrm{E}}}{}^{\dot{\mathrm{C}}}, \tag{13}$$

which, together with (12), is equivalent to the tensorial relation (11) due to the connection of L with $\mathrm{A} = (\mathrm{A}^{\mathrm{B}}{}_{\mathrm{C}})$ and $\mathrm{A}^{\dagger} = \left(\mathrm{A}^{\dagger}{}_{\dot{\mathrm{B}}}{}^{\dot{\mathrm{C}}}\right)$.

The paper is organized as follows. In Sec. 2 it is undertaken a detailed study of the spinor X, which allows to introduce in a natural manner the Infeld-van der Waerden symbols $\sigma_{\mu}{}^{\mathrm{A}\dot{\mathrm{B}}}$ [30, 31], of great importance to perform the spinor transcription of a tensor expression, or analogously, given a certain tensor, to deduce its corresponding spinor. We show that these symbols provide an explicit formula for L in terms of A and $\mathrm{A}^{\dagger}$, in harmony with the results obtained in [32]. On the other hand, it is established that for $x^{\mu}$ null it is possible to express $\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}$ as the product of simple spinors, which turns out to be relevant in the spinorial analysis of vectors on the light cone. In Sec. 3, we consider the simple spinors associated with an arbitrary null tetrad of Newman-Penrose (NP) type [33], which in turn generates spinors for a real orthonormal Minkowskian tetrad, that facilitates the spinorial study of any tensor (for instance, the skew-symmetric tensor of the Maxwell field) written in terms of a real tetrad, or in terms of a NP type. In [34-37], it was used a real tetrad to build a basis for any skew-symmetric tensor of second order, with the aim of analyzing the trajectories of charged particles (with or without radiation reaction) in special relativity. Finally, in Sec. 4 we apply this technique to the electromagnetic tensor, and it is shown the existence of the Maxwell symmetric spinor. The method and the results of this section are successfully applied to obtain the spinorial structure of Maxwell's energy-momentum tensor. In addition, in Secs. 2-4 we also indicate the NP versions of the main spinorial relations.

## 2. THE CARTAN SPINOR

In (12) it is immediate the following expansion:

$$X = \frac{x^0}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{x^1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \frac{x^2}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{x^3}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad (14)$$

which motivates the introduction of the Infeld-van der Waerden symbols [30, 31, 38, 39] in terms of the Cayley-Sylvester-Pauli matrices indicated in (5):

$$(\sigma^0{}_{\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\,\mathrm{I}\,, \qquad (\sigma^j{}_{\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\,\sigma_j\,, \qquad j = 1, 2, 3, \qquad (15)$$

notice that it is verified the Hermitian property $\overline{\sigma_\mu{}^{\mathrm{A\dot{B}}}} = \sigma_\mu{}^{\mathrm{B\dot{A}}}$. Then (14) acquires the form:

$$\mathrm{X}^{\mathrm{A\dot{B}}} = x^\mu\,\sigma_\mu{}^{\mathrm{A\dot{B}}}\,, \qquad x^\mu \;\leftrightarrow\; \mathrm{X}^{\mathrm{A\dot{B}}}\,, \qquad (16)$$

whose structure shows the pattern to follow for constructing the 2-spinor associated with a vector, for each tensor index we have a pair of spinor indices. In $\sigma_\mu{}^{\mathrm{A\dot{B}}}$ the $\mu$ index can be raised with the Minkowski metric $(g_{\mu\nu}) = Diag\,(1, -1, -1, -1)$, thus:

$$(\sigma_{0\,\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\,\mathrm{I}\,, \qquad (\sigma_{j\,\mathrm{A\dot{B}}}) = -\frac{1}{\sqrt{2}}\,\sigma_k\,, \qquad k = 1, 2, 3, \qquad (17)$$

then it is easy to prove that:

$$\sigma^{\mu\,\mathrm{A\dot{B}}}\,\sigma_\mu{}^{\mathrm{C\dot{D}}} = \epsilon^{\mathrm{AC}}\,\epsilon^{\mathrm{\dot{B}\dot{D}}}\,, \qquad (18)$$

with the skew-symmetric matrices:

$$(\epsilon^{\mathrm{AB}}) = (\epsilon_{\mathrm{AB}}) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad (\epsilon^{\mathrm{\dot{A}\dot{B}}}) = (\epsilon_{\mathrm{\dot{A}\dot{B}}}) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \epsilon_{\mathrm{AB}}\,\epsilon^{\mathrm{CD}} = \delta^{\mathrm{C}}_{\mathrm{A}}\,\delta^{\mathrm{D}}_{\mathrm{B}} - \delta^{\mathrm{D}}_{\mathrm{A}}\delta^{\mathrm{C}}_{\mathrm{B}}\,,$$

$$\epsilon_{\mathrm{B}}{}^{\mathrm{D}} = -\epsilon^{\mathrm{D}}{}_{\mathrm{B}} = \epsilon_{\mathrm{AB}}\,\epsilon^{\mathrm{AD}} = \delta^{\mathrm{D}}_{\mathrm{B}}\,, \qquad \epsilon_{\mathrm{AB}}\,\epsilon^{\mathrm{AB}} = 2\,; \qquad (19)$$

in [40] it is indicated that $\epsilon_{\mathrm{AC}}$ is the quantity that defines the sympletic complex structure of the spin space.

The spinorial indices can be raised and lowered by means of (19) (we shall employ the Penrose-Rindler convention [41]):

$$\psi^{\mathrm{A}} = \epsilon^{\mathrm{AB}}\,\psi_{\mathrm{B}}\,, \qquad \psi_{\mathrm{C}} = \epsilon_{\mathrm{BC}}\,\psi^{\mathrm{B}}\,, \qquad \psi^{\mathrm{\dot{A}}} = \epsilon^{\mathrm{\dot{A}\dot{B}}}\,\psi_{\mathrm{\dot{B}}}\,, \qquad \psi_{\mathrm{\dot{C}}} = \epsilon_{\mathrm{\dot{B}\dot{C}}}\,\psi^{\mathrm{\dot{B}}}\,, \qquad (20)$$

that is, for an arbitrary simple spinor:

$$\psi^1 = \psi_2\,, \quad \psi^2 = -\psi_1 \quad \therefore \quad \psi^{\mathrm{A}}\psi_{\mathrm{A}} = 0\,, \quad \psi^{\mathrm{\dot{A}}}\psi_{\mathrm{\dot{A}}} = 0\,, \quad \psi^{\mathrm{A}}\phi_{\mathrm{A}} = -\psi_{\mathrm{A}}\phi^{\mathrm{A}}\,. \qquad (21)$$

Furthermore,

$$(\sigma^0{}_{\mathrm{A\dot{B}}}) = (\sigma_{0\,\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\,\mathrm{I}\,, \qquad (\sigma^1{}_{\mathrm{A\dot{B}}}) = (-\sigma_{1\,\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\sigma_1\,,$$
$$(\sigma^2{}_{\mathrm{A\dot{B}}}) = (-\sigma_{2\,\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\sigma_2\,, \qquad (\sigma^3{}_{\mathrm{A\dot{B}}}) = (-\sigma_{3\,\mathrm{A\dot{B}}}) = \frac{1}{\sqrt{2}}\sigma_3\,, \tag{22}$$

and together with (15) it implies the interesting relation [42]:

$$\sigma_\mu{}^{\mathrm{A\dot{B}}}\,\sigma_{\nu\,\mathrm{A\dot{B}}} = g_{\mu\nu}\,, \tag{23}$$

that allows to invert the equation (16):

$$x^\mu = \sigma^\mu{}_{\mathrm{A\dot{B}}}\,\mathrm{X}^{\mathrm{A\dot{B}}}\,, \tag{24}$$

where it is shown how to obtain the vector associated with a 2-spinor, indeed, the Infeld-van der Waerden symbols capture one pair of spinor indices to assign one tensor index.

The numerical values (15) are not altered with a change of reference frame:

$$\sigma^{\mu\,\mathrm{B\dot{C}}} = \mathrm{L}^\mu{}_\nu\,\mathrm{A}^{\mathrm{B}}{}_{\mathrm{D}}\,\sigma^{\nu\,\mathrm{D\dot{E}}}\,\mathrm{A}^\dagger{}_{\dot{\mathrm{E}}}{}^{\dot{\mathrm{C}}}\,, \tag{25}$$

it is not difficult to invert the matrices (6) and (8), see [41]:

$$(\mathrm{A}^{-1\ \mathrm{B}}{}_{\mathrm{C}}) = \begin{pmatrix} a_0 - ia_3 & a_2 - ia_1 \\ -a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}, \quad \left(\mathrm{A}^{\dagger\,-1}{}_{\dot{\mathrm{B}}}{}^{\dot{\mathrm{C}}}\right) = \begin{pmatrix} a_0^* + ia_3^* & -a_2^* + ia_1^* \\ a_2^* + ia_1^* & a_0^* - ia_3^* \end{pmatrix}, \tag{26}$$

then, using (23) and (26) in (25) the Lorentz matrix is deduced in terms of the Olinde Rodrigues parameters and of the Infeld-van der Waerden symbols:

$$\mathrm{L}^\mu{}_\nu = \sigma^\mu{}_{\mathrm{D\dot{E}}}\ \sigma_\nu{}^{\mathrm{B\dot{C}}}\ \mathrm{A}^{-1\,\mathrm{D}}{}_{\mathrm{B}}\ \mathrm{A}^{\dagger\,-1}{}_{\dot{\mathrm{C}}}{}^{\dot{\mathrm{E}}}\,, \tag{27}$$

from which the expressions of [32] are immediate and the explicit formulas of [22, 25, 26] for L in terms of the aforesaid parameters.

With the prescription (16) and the identity (18) we can build the spinor associated with the metric tensor:

$$g_{\mathrm{AC\dot{B}\dot{D}}} = \sigma^\mu{}_{\mathrm{A\dot{B}}}\,g_{\mu\nu}\,\sigma^\nu{}_{\mathrm{C\dot{D}}} = \sigma^\mu{}_{\mathrm{A\dot{B}}}\,\sigma_{\mu\,\mathrm{C\dot{D}}} = \epsilon_{\mathrm{AC}}\,\epsilon_{\mathrm{\dot{B}\dot{D}}}\,, \tag{28}$$

compatible with (23), and from (24) we obtain $x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} = \mathrm{X}^{\mathrm{A\dot{B}}}\mathrm{X}^{\mathrm{C\dot{D}}}\epsilon_{\mathrm{AC}}\epsilon_{\mathrm{\dot{B}\dot{D}}}$, therefore:

$$x^\mu x_\mu = \mathrm{X}^{\mathrm{A\dot{B}}}\mathrm{X}_{\mathrm{A\dot{B}}}\,. \tag{29}$$

If the 2-spinor $\mathrm{X}^{\mathrm{D\dot{E}}}$ is the product of two simple spinors:

$$\mathrm{X}^{\mathrm{A\dot{B}}} = \xi^{\mathrm{A}}\,\xi^{\mathrm{\dot{B}}}, \quad \mathrm{X}_{2x2} = (\xi^{\mathrm{A}})_{2x1}\,(\xi^{\mathrm{\dot{B}}})_{1x2}, \; (\xi^{\mathrm{A}}) = \begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}, \quad (\xi^{\mathrm{\dot{B}}}) = (\xi^{\dot{1}} \;\; \xi^{\dot{2}}) = (\overline{\xi^1} \;\; \overline{\xi^2}) = \begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}^{\dagger}, \tag{30}$$

then, its associated vector must be null because with (21) and (30) we have that $\mathrm{X}^{\mathrm{A\dot{B}}}\,\mathrm{X}_{\mathrm{A\dot{B}}} = \left|\xi^{\mathrm{A}}\xi_{\mathrm{A}}\right|^2 = 0$, thus (29) gives:

$$c^2t^2 - x^2 - y^2 - z^2 = x^{\mu}x_{\mu} = 0\,, \tag{31}$$

$x^{\nu}$ is on a light cone, and it can be assumed that it is pointing to the future $(x^0 > 0)$. Now, the aim is to calculate $\xi^{\mathrm{A}}$ given $\mathrm{X}^{\mathrm{A\dot{B}}}$:

$$\xi^1 = p\,e^{i\varphi}\,, \qquad \xi^{\dot{1}} = p\,e^{-i\varphi}\,, \qquad \xi^2 = q\,e^{i\theta}\,, \qquad \xi^{\dot{2}} = q\,e^{-i\theta}\,, \tag{32}$$

and the expressions:

$$p^2 = \xi^1\,\xi^{\dot{1}} = \mathrm{X}^{1\dot{1}} = \tfrac{1}{\sqrt{2}}(x^0 + x^3), \qquad q^2 = \xi^2\,\xi^{\dot{2}} = \mathrm{X}^{2\dot{2}} = \tfrac{1}{\sqrt{2}}(x^0 - x^3), \tag{33}$$

are univocally determining the magnitudes $p$ and $q$, however, $\theta$ and $\varphi$ are arbitrary except for their difference which can be obtained by means of:

$$pq\,e^{i(\varphi-\theta)} = \tfrac{1}{\sqrt{2}}(x^1 + ix^2), \tag{34}$$

this manifests the non-unicity of $\xi^{\mathrm{A}}$ due to the fact that it can be multiplied by an arbitrary phase without altering the Cartan 2-spinor:

$$\mathrm{X}^{\mathrm{A\dot{B}}} = \xi^{\mathrm{A}}\,\xi^{\mathrm{\dot{B}}} = \xi^{\mathrm{A}}\,\overline{\xi^{\mathrm{B}}} = \left(e^{i\Omega}\,\xi^{\mathrm{A}}\right)\left(\overline{e^{i\Omega}\,\xi^{\mathrm{B}}}\right). \tag{35}$$

If $x^{\mu}$ points to the past $(x^0 < 0)$, the decomposition will take the form $\mathrm{X}^{\mathrm{A\dot{B}}} = -\xi^{\mathrm{A}}\,\xi^{\mathrm{\dot{B}}}$. In summary, if $k^{\nu}$ is a real null vector, then:

$$k^{\mu} \qquad \leftrightarrow \qquad \mathrm{K}^{\mathrm{A\dot{B}}} = \gamma^{\mathrm{A}}\,\gamma^{\mathrm{\dot{B}}}\,, \tag{36}$$

where $\gamma^{\mathrm{C}}$ is defined up to an arbitrary phase.

With (36) and $t^{\mu} \;\leftrightarrow\; \mathrm{T}^{\mathrm{A\dot{B}}} = \eta^{\mathrm{A}}\,\eta^{\mathrm{\dot{B}}}$ one gets the inner product:

$$k^{\mu}\,t_{\mu} = \left|\gamma_{\mathrm{A}}\eta^{\mathrm{A}}\right|^2, \tag{37}$$

therefore $k^{\nu}t_{\nu} = 0$ if and only if $\gamma_{\mathrm{A}}\eta^{\mathrm{A}} = 0$, but Synge [22] demonstrates that $k^{\nu}t_{\nu} = 0$ implies the proportionality of such null vectors, thus:

$$k^\mu = \lambda\, t^\mu \qquad \Leftrightarrow \qquad \gamma_{\rm A}\eta^{\rm A} = 0, \tag{38}$$

in whose case $\gamma_{\rm A} = \sqrt{\lambda}\,\eta_{\rm A}$. When studying the Newman-Penrose tetrad and the electromagnetic field, a situation with $k^\mu t_\mu = 1$ arises, and the arbitrarity in the phases of $\gamma^{\rm A}$ and $\eta^{\rm B}$ allows to choose the norm $\gamma_{\rm A}\eta^{\rm A} = -\gamma^{\rm A}\eta_{\rm A} = 1$, in harmony with (37).

In Sec. 3, the analysis made here (for the Cartan $X^{A\dot{B}}$) is extended to a null tetrad of the NP type [33] and to its corresponding real orthonormal tetrad, which is important in the spinorial structure of the Faraday and Maxwell tensors (Sec. 4).

## 3. TETRADS AND THEIR 2-SPINORS

For each event in the spacetime it can be constructed a real orthonormal tetrad:

$$\mathrm{e}_{(0)}{}^{\mu}\, \mathrm{e}_{(0)\,\mu} = 1, \qquad \mathrm{e}_{(0)}{}^{\mu}\, \mathrm{e}_{(j)\,\mu} = 0, \qquad \mathrm{e}_{(j)}{}^{\nu}\, \mathrm{e}_{(k)\,\nu} = -\delta_{jk}, \qquad j,k = 1,2,3 \tag{39}$$

positive-oriented:

$$\eta_{\mu\nu\alpha\beta}\, \mathrm{e}_{(0)}{}^{\mu}\, \mathrm{e}_{(1)}{}^{\nu}\, \mathrm{e}_{(2)}{}^{\alpha}\, \mathrm{e}_{(3)}{}^{\beta} = 1, \tag{40}$$

where the totally skew-symmetric Levi-Civita tensor takes part:

$$\eta^{\mu\nu\alpha\beta} = -\eta_{\mu\nu\alpha\beta} = -1 \text{ or } 1 \text{ if } (\mu\nu\alpha\beta) \text{ is even or odd permutation of (0123), respectively, and 0 if two of its indices have the same value.} \tag{41}$$

The real tetrad permits to establish a basis for any tensorial object, for example, the electromagnetic field tensor (Sec. 4), thus the spinorial study of (39) is useful in the deduction of the Maxwell spinor and does also provide a convenient platform for the spinor formulation of differential geometry of curves [43, 44]; besides, it leads to the Newman-Penrose null tetrad [33, 45-47]:

$$l^\mu = \frac{1}{\sqrt{2}}\left(\mathrm{e}_{(0)}{}^{\mu} + \mathrm{e}_{(3)}{}^{\mu}\right), n^\mu = \frac{1}{\sqrt{2}}\left(\mathrm{e}_{(0)}{}^{\mu} - \mathrm{e}_{(3)}{}^{\mu}\right), m^\mu = \frac{1}{\sqrt{2}}\left(\mathrm{e}_{(1)}{}^{\mu} - i\mathrm{e}_{(2)}{}^{\mu}\right), \overline{m}^\mu = \frac{1}{\sqrt{2}}\left(\mathrm{e}_{(1)}{}^{\mu} + i\mathrm{e}_{(2)}{}^{\mu}\right), \tag{42}$$

with the properties:

$$l^\mu n_\mu = -m^\nu \overline{m}_\nu = 1, \qquad l^\mu m_\mu = n^\nu m_\nu = 0, \qquad l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = 0, \qquad \eta_{\mu\nu\alpha\beta} l^\mu n^\nu m^\alpha \overline{m}^\beta = -i. \tag{43}$$

According to (36), the real null vectors $n^\mu$ and $l^\mu$ have got associated simple spinors, which we shall denote with the Greek letters iota and omicron, respectively:

$$l^{\rm A\dot{B}} = o^{\rm A} o^{\rm \dot{B}}, \qquad n^{\rm A\dot{B}} = \iota^{\rm A} \iota^{\rm \dot{B}}, \qquad o_{\rm A}\iota^{\rm A} = -o^{\rm A}\iota_{\rm A} = 1\,. \tag{44}$$

The vector $m^{\mu}$ is associated with the spinor $m^{\mathrm{A}\dot{\mathrm{B}}}$ that can be written in terms of a basis of simple spinors:

$$m^{\mathrm{C}\dot{\mathrm{D}}} = \lambda_1\, o^{\mathrm{C}} o^{\dot{\mathrm{D}}} + \lambda_2\, \iota^{\mathrm{C}} \iota^{\dot{\mathrm{D}}} + \lambda_3\, o^{\mathrm{C}} \iota^{\dot{\mathrm{D}}} + \lambda_4\, \iota^{\mathrm{C}} o^{\dot{\mathrm{D}}}\,,$$

but (43) imposes conditions, for example, $o_{\mathrm{C}} o_{\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ implies $\lambda_2 = 0$, $\iota_{\mathrm{C}} \iota_{\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ gives $\lambda_1 = 0$, $m_{\mathrm{C}\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ leads to $\lambda_3 \lambda_4 = 0$, and $m_{\mathrm{C}\dot{\mathrm{D}}} \overline{m^{\mathrm{D}\dot{\mathrm{C}}}} = -1$ requires that $|\lambda_3|^2 + |\lambda_4|^2 = 1$, then without loss of generality we choose $\lambda_3 = 1$ and $\lambda_4 = 0$, therefore:

$$m^{\nu} \;\leftrightarrow\; m^{\mathrm{C}\dot{\mathrm{D}}} = o^{\mathrm{C}} \iota^{\dot{\mathrm{D}}}\,, \qquad \bar{m}^{\nu} \;\leftrightarrow\; \mathrm{M}^{\mathrm{C}\dot{\mathrm{D}}} = \overline{m^{\mathrm{D}\dot{\mathrm{C}}}} = \iota^{\mathrm{C}} o^{\dot{\mathrm{D}}}\,, \tag{45}$$

which together with (42) and (44) gives the spinors associated with the real tetrad:

$$\begin{aligned} \mathrm{e}_{(0)\,\nu} \;&\leftrightarrow\; \frac{1}{\sqrt{2}}(o_{\mathrm{A}} o_{\dot{\mathrm{B}}} + \iota_{\mathrm{A}} \iota_{\dot{\mathrm{B}}}), \qquad \mathrm{e}_{(1)\,\nu} \;\leftrightarrow\; \frac{1}{\sqrt{2}}(o_{\mathrm{A}} \iota_{\dot{\mathrm{B}}} + \iota_{\mathrm{A}} o_{\dot{\mathrm{B}}}), \\ \mathrm{e}_{(2)\,\nu} \;&\leftrightarrow\; \frac{i}{\sqrt{2}}(o_{\mathrm{A}} \iota_{\dot{\mathrm{B}}} - \iota_{\mathrm{A}} o_{\dot{\mathrm{B}}}), \qquad \mathrm{e}_{(3)\,\nu} \;\leftrightarrow\; \frac{1}{\sqrt{2}}(o_{\mathrm{A}} o_{\dot{\mathrm{B}}} - \iota_{\mathrm{A}} \iota_{\dot{\mathrm{B}}}), \end{aligned} \tag{46}$$

consistent with (39), and taking into account the norm indicated in (44).

With the real and NP tetrads, it is straightforward to generate the metric tensor:

$$g_{\mu\nu} = l_{\mu} n_{\nu} + l_{\nu} n_{\mu} - m_{\mu} \bar{m}_{\nu} - m_{\nu} \bar{m}_{\mu} = \mathrm{e}_{(0)\,\mu} \mathrm{e}_{(0)\,\nu} - \mathrm{e}_{(j)\,\mu} \mathrm{e}_{(j)\,\nu}\,, \qquad j = 1, 2, 3 \tag{47}$$

where we can use (44) and (45) or (46) to deduce the spinor associated with the Minkowski metric:

$$g_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = (o_{\mathrm{A}} \;\mathrm{x}\; \iota_{\mathrm{C}})\,(o_{\dot{\mathrm{B}}} \;\mathrm{x}\; \iota_{\dot{\mathrm{D}}}), \tag{48}$$

with the Lowry notation [48] applicable to tensorial and spinorial indices:

$$\mathrm{A}_{\mu} \;\mathrm{x}\; \mathrm{B}_{\nu} \;\equiv\; \mathrm{A}_{\mu}\, \mathrm{B}_{\nu} - \mathrm{A}_{\nu}\, \mathrm{B}_{\mu}\,, \tag{49}$$

and comparing this with (28) we get the expressions:

$$\epsilon_{\mathrm{AB}} = o_{\mathrm{A}} \;\mathrm{x}\; \iota_{\mathrm{B}} = o_{\mathrm{A}} \iota_{\mathrm{B}} - o_{\mathrm{B}} \iota_{\mathrm{A}}\,, \qquad \epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}} = o_{\dot{\mathrm{B}}} \;\mathrm{x}\; \iota_{\dot{\mathrm{D}}}\,, \tag{50}$$

which are valid for any pair of simple spinors that fulfill the normalization (44); in particular, by means of (50) it is immediate to obtain the useful relation:

$$2(o_{\mathrm{A}} \iota_{\mathrm{C}} o_{\dot{\mathrm{B}}} \iota_{\dot{\mathrm{D}}} - o_{\mathrm{C}} \iota_{\mathrm{A}} o_{\dot{\mathrm{D}}} \iota_{\dot{\mathrm{B}}}) = \epsilon_{\mathrm{AC}}\,(o_{\dot{\mathrm{B}}} \iota_{\dot{\mathrm{D}}} + o_{\dot{\mathrm{D}}} \iota_{\dot{\mathrm{B}}}) + (o_{\mathrm{A}} \iota_{\mathrm{C}} + o_{\mathrm{C}} \iota_{\mathrm{A}})\,\epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}} \tag{51}$$

The real tetrad gives rise to the following six skew-symmetric tensors, which are quite relevant in the study of the movement of classical charged particles [34-37, 49]:

$$W_{(j)\,\mu\nu} = e_{(0)\,\mu} \text{ x } e_{(j)\,\nu}\,, \quad j = 1, 2, 3, \qquad W_{(4)\,\mu\nu} = e_{(1)\,\mu} \text{ x } e_{(2)\,\nu}\,,$$
$$W_{(5)\,\mu\nu} = e_{(1)\,\mu} \text{ x } e_{(3)\,\nu}\,, \qquad W_{(6)\,\mu\nu} = e_{(2)\,\mu} \text{ x } e_{(3)\,\nu}\,, \tag{52}$$

and from (39) (without sum over $r$):

$$W_{(r)\,\mu\nu}\, W_{(r)}{}^{\mu\nu} = -2 \text{ or } 2 \quad \text{for} \quad r = 1, 2, 3 \quad \text{or} \quad r = 4, 5, 6, \text{ respectively},$$
$$W_{(j)\,\mu\nu}\, W_{(k)}{}^{\mu\nu} = 0, \quad j \neq k\,, \quad j, k = 1, 2, \ldots, 6\,. \tag{53}$$

The concept of dual tensor [22, 50-53]:

$$^{+}F_{\mu\nu} = \frac{1}{2}\, \eta_{\mu\nu\alpha\beta}\, F^{\alpha\beta}, \qquad F_{\mu\nu} = -F_{\nu\mu}\,, \tag{54}$$

together with (52), implies the connection:

$$^{+}W_{(n)\,\mu\nu} = -(-1)^{n}\, W_{(7-n)\,\mu\nu}\,, \qquad n = 1, 2, \ldots, 6\,, \tag{55}$$

namely, $^{+}W_{(2)\,\alpha\beta} = -W_{(5)\,\alpha\beta}$, $^{+}W_{(3)\,\alpha\beta} = W_{(4)\,\alpha\beta}$, etc., with importance in the study of the algebraic composition of the Faraday tensor.

By means of (46) and (51) it is obtained the spinorial version of (52):

$$\begin{aligned}
W_{(1)\,AC\dot{B}\dot{D}} &= \frac{1}{2}[(o_A o_C - \iota_A \iota_C)\, \epsilon_{\dot{B}\dot{D}} + \epsilon_{AC}\, (o_{\dot{B}} o_{\dot{D}} - \iota_{\dot{B}} \iota_{\dot{D}})],\\
W_{(2)\,AC\dot{B}\dot{D}} &= \frac{i}{2}[(o_A o_C + \iota_A \iota_C)\, \epsilon_{\dot{B}\dot{D}} - \epsilon_{AC}\, (o_{\dot{B}} o_{\dot{D}} + \iota_{\dot{B}} \iota_{\dot{D}})],\\
W_{(3)\,AC\dot{B}\dot{D}} &= -\frac{1}{2}[(o_A \iota_C + o_C \iota_A)\, \epsilon_{\dot{B}\dot{D}} + \epsilon_{AC}\, (o_{\dot{B}} \iota_{\dot{D}} + o_{\dot{D}} \iota_{\dot{B}})],\\
W_{(4)\,AC\dot{B}\dot{D}} &= \frac{i}{2}[(o_A \iota_C + o_C \iota_A)\, \epsilon_{\dot{B}\dot{D}} - \epsilon_{AC}\, (o_{\dot{B}} \iota_{\dot{D}} + o_{\dot{D}} \iota_{\dot{B}})],\\
W_{(5)\,AC\dot{B}\dot{D}} &= -\frac{1}{2}[(o_A o_C + \iota_A \iota_C)\, \epsilon_{\dot{B}\dot{D}} + \epsilon_{AC}\, (o_{\dot{B}} o_{\dot{D}} + \iota_{\dot{B}} \iota_{\dot{D}})],\\
W_{(6)\,AC\dot{B}\dot{D}} &= \frac{i}{2}[-(o_A o_C - \iota_A \iota_C)\, \epsilon_{\dot{B}\dot{D}} + \epsilon_{AC}\, (o_{\dot{B}} o_{\dot{D}} - \iota_{\dot{B}} \iota_{\dot{D}})],
\end{aligned} \tag{56}$$

where it is verified the property $\overline{W_{(r)\,AC\dot{B}\dot{D}}} = W_{(r)\,BD\dot{A}\dot{C}}$ because the $W_{(r)\,\mu\nu}$ are real. The relations given by (56) motivate the following comment of Rindler [20]:

'Every spinor can be written as a linear combination of symmetric spinors multiplied by $\epsilon_{AB}$ or/and $\epsilon_{\dot{A}\dot{B}}$'. (57)

In the structure of (56) we can observe the repetition of different kinds of terms, then it is natural to introduce the spinors:

$$V_{AC\dot{B}\dot{D}} = o_A o_C\, \epsilon_{\dot{B}\dot{D}}\,, \qquad U_{AC\dot{B}\dot{D}} = \iota_A \iota_C\, \epsilon_{\dot{B}\dot{D}}\,, \qquad M_{AC\dot{B}\dot{D}} = -(o_A \iota_C + o_C \iota_A)\, \epsilon_{\dot{B}\dot{D}}\,, \tag{58}$$

and the prescription (24), together with (44), (45) and (50), gives their tensorial counterpart:

$$\mathrm{V}_{\mu\nu} = l_\mu \,\mathrm{x}\, m_\nu\,, \qquad \mathrm{U}_{\mu\nu} = \bar{m}_\mu \,\mathrm{x}\, n_\nu\,, \qquad \mathrm{M}_{\mu\nu} = m_\mu \,\mathrm{x}\, \bar{m}_\nu + n_\mu \,\mathrm{x}\, l_\nu\,, \tag{59}$$

with importance in the formalism of NP [33, 45-47].

The Levi-Civita tensor admits a representation in terms of the real and NP tetrads:

$$\eta_{\mu\nu\alpha\beta} = -\begin{vmatrix} \mathrm{e}_{(0)\mu} & \mathrm{e}_{(0)\nu} & \mathrm{e}_{(0)\alpha} & \mathrm{e}_{(0)\beta} \\ \mathrm{e}_{(1)\mu} & \mathrm{e}_{(1)\nu} & \mathrm{e}_{(1)\alpha} & \mathrm{e}_{(1)\beta} \\ \mathrm{e}_{(2)\mu} & \mathrm{e}_{(2)\nu} & \mathrm{e}_{(2)\alpha} & \mathrm{e}_{(2)\beta} \\ \mathrm{e}_{(3)\mu} & \mathrm{e}_{(3)\nu} & \mathrm{e}_{(3)\alpha} & \mathrm{e}_{(3)\beta} \end{vmatrix} = -i \begin{vmatrix} l_\mu & l_\nu & l_\alpha & l_\beta \\ n_\mu & n_\nu & n_\alpha & n_\beta \\ m_\mu & m_\nu & m_\alpha & m_\beta \\ \bar{m}_\mu & \bar{m}_\nu & \bar{m}_\alpha & \bar{m}_\beta \end{vmatrix}, \tag{60}$$

where the positive orientation indicated by (40) and (43) is respected. With (52), (54) and (60) it is easy to prove (55); besides, (60) leads to the relations:

$$\eta_{\mu\nu\alpha\beta}\, l^\alpha m^\beta = -i\, l_\mu \,\mathrm{x}\, m_\nu\,, \qquad \eta_{\mu\nu\alpha\beta}\, m^\alpha \bar{m}^\beta = i\, l_\mu \,\mathrm{x}\, n_\nu\,,$$
$$\eta_{\mu\nu\alpha\beta}\, n^\alpha m^\beta = i\, n_\mu \,\mathrm{x}\, m_\nu\,, \qquad \eta_{\mu\nu\alpha\beta}\, l^\alpha n^\beta = i\, m_\mu \,\mathrm{x}\, \bar{m}_\nu\,, \tag{61}$$

which imply the self-dual character of (59):

$$^{+}\mathrm{V}_{\mu\nu} = -i\, \mathrm{V}_{\mu\nu}\,, \qquad ^{+}\mathrm{U}_{\mu\nu} = -i\, \mathrm{U}_{\mu\nu}\,, \qquad ^{+}\mathrm{M}_{\mu\nu} = -i\, \mathrm{M}_{\mu\nu}\,. \tag{62}$$

Projecting (60) onto the Infeld-van der Waerden symbols [54], and employing (44), (45) and (50), we obtain the corresponding spinor [39]:

$$\eta_{\mathrm{ACEG}\dot{\mathrm{B}}\dot{\mathrm{D}}\dot{\mathrm{F}}\dot{\mathrm{H}}} = i\,(\epsilon_{\mathrm{AE}}\, \epsilon_{\mathrm{CG}}\, \epsilon_{\dot{\mathrm{B}}\dot{\mathrm{H}}}\, \epsilon_{\dot{\mathrm{D}}\dot{\mathrm{F}}} - \epsilon_{\mathrm{AG}}\, \epsilon_{\mathrm{CE}}\, \epsilon_{\dot{\mathrm{B}}\dot{\mathrm{F}}}\, \epsilon_{\dot{\mathrm{D}}\dot{\mathrm{H}}}). \tag{63}$$

Let $\mathrm{F}_{\mu\nu}$ be an arbitrary tensor, then the prescription (16) gives its associated spinor:

$$\mathrm{F}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = \sigma^{\mu}{}_{\mathrm{A}\dot{\mathrm{B}}}\, \mathrm{F}_{\mu\nu}\, \sigma^{\nu}{}_{\mathrm{C}\dot{\mathrm{D}}}\,, \tag{64}$$

from where:

$$\overline{\mathrm{F}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}}} = \mathrm{F}_{\mathrm{BD}\dot{\mathrm{A}}\dot{\mathrm{C}}} \quad \text{because } \mathrm{F}_{\alpha\beta} \text{ is real,}$$
$$\mathrm{F}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = -\mathrm{F}_{\mathrm{CA}\dot{\mathrm{D}}\dot{\mathrm{B}}} \quad \text{because } \mathrm{F}_{\mu\nu} \text{ is skew-symmetric,} \tag{65}$$

these results together with (54) and (63) lead to the spinor associated with the dual tensor:

$$^{+}\mathrm{F}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = i\, \mathrm{F}_{\mathrm{AC}\dot{\mathrm{D}}\dot{\mathrm{B}}}\,, \tag{66}$$

which also verifies the symmetries (65).

Penrose [40] asseverates that the 2-spinors formalism is not only simpler when it comes to establish properties of conformal invariance, but does also provide a more systematical overview when it comes to understand the propagation of massless fields. Then, Sec. 4 is devoted to the study of some spinorial aspects of the electromagnetic field.

## 4. THE FARADAY TENSOR AND MAXWELL SPINOR

Minkowski [55] introduced the skew-symmetric tensor:

$$(\mathrm{F}^{\mu\nu}) = \begin{pmatrix} 0 & -\mathrm{E}_1 & -\mathrm{E}_2 & -\mathrm{E}_3 \\ \mathrm{E}_1 & 0 & -c\mathrm{B}_3 & c\mathrm{B}_2 \\ \mathrm{E}_2 & c\mathrm{B}_3 & 0 & -c\mathrm{B}_1 \\ \mathrm{E}_3 & -c\mathrm{B}_2 & c\mathrm{B}_1 & 0 \end{pmatrix}, \tag{67}$$

where $\vec{\mathrm{E}} = (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ and $\vec{\mathrm{B}} = (\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3)$ are the electric and magnetic fields expressed in the MKS system of units, respectively. With (54) it is obtained the dual tensor:

$$({}^{*}\mathrm{F}^{\mu\nu}) = \begin{pmatrix} 0 & c\mathrm{B}_1 & c\mathrm{B}_2 & c\mathrm{B}_3 \\ -c\mathrm{B}_1 & 0 & -\mathrm{E}_3 & \mathrm{E}_2 \\ -c\mathrm{B}_2 & \mathrm{E}_3 & 0 & -\mathrm{E}_1 \\ -c\mathrm{B}_3 & -\mathrm{E}_2 & \mathrm{E}_1 & 0 \end{pmatrix}, \tag{68}$$

this matrix arises directly from (67) making the substitutions:

$$\vec{\mathrm{E}} \longrightarrow -c\vec{\mathrm{B}}, \qquad c\vec{\mathrm{B}} \longrightarrow \vec{\mathrm{E}}, \tag{69}$$

which is a symmetry of the vacuum Maxwell equations:

$$\vec{\nabla}\cdot\vec{\mathrm{E}} = \vec{\nabla}\cdot\vec{\mathrm{B}} = 0, \qquad \vec{\nabla}\,\mathrm{x}\,\vec{\mathrm{E}} = -\frac{\partial\vec{\mathrm{B}}}{\partial t}, \qquad \vec{\nabla}\,\mathrm{x}\,\vec{\mathrm{B}} = \frac{1}{\mathrm{c}^2}\frac{\partial\vec{\mathrm{E}}}{\partial t}, \tag{70}$$

this is because they are invariant under duality rotations [4, 51, 56-59], indeed, if $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ satisfy (70) then the fields:

$$\vec{\mathrm{E}}' = \vec{\mathrm{E}}\cos\xi + c\,\vec{\mathrm{B}}\,\mathrm{sen}\,\xi, \qquad c\,\vec{\mathrm{B}}' = -\vec{\mathrm{E}}\,\mathrm{sen}\,\xi + c\,\vec{\mathrm{B}}\cos\xi, \tag{71}$$

also fulfill such equations; the mapping (69) is a duality rotation with $\xi = -\pi/2$. The transformation (71) motivates the use of the Riemann [60]-Silberstein [61, 62] complex vector (thus named by Bialynicki-Birula [63]) [64]:

$$\vec{\mathrm{F}} = c\,\vec{\mathrm{B}} + i\,\vec{\mathrm{E}}, \tag{72}$$

therefore, the duality rotations correspond to a phase transformation:

$$\vec{\mathrm{F}}' = e^{i\xi}\,\vec{\mathrm{F}}. \tag{73}$$

The field equations (70) can be obtained by means of a variational principle [2] whose Lagrangian function is invariant under (71), then the Noether's theorem [2, 65-71] implies the conservation of the electromagnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\,\mathrm{E}^2 + \frac{1}{2\mu_0}\,\mathrm{B}^2\right) + \vec{\nabla}\cdot\left(\frac{1}{\mu_0}\,\vec{\mathrm{E}}\times\vec{\mathrm{B}}\right) = 0\,, \tag{74}$$

given in terms of the energy density and Poynting's vector [72]. On the other hand, using (71) into the matrix (67) and its dual, we deduce the transformations:

$$\mathrm{F}'^{\mu\nu} = \mathrm{F}^{\mu\nu}\cos\xi - {}^{+}\mathrm{F}^{\mu\nu}\,\mathrm{sen}\,\xi\,, \qquad {}^{+}\mathrm{F}'^{\mu\nu} = \mathrm{F}^{\mu\nu}\,\mathrm{sen}\,\xi + {}^{+}\mathrm{F}^{\mu\nu}\cos\xi\,, \tag{75}$$

these relations make it convenient to introduce the complex Faraday tensor [73]:

$$\mathrm{W}_{\mu\nu} = \mathrm{F}_{\mu\nu} + i\,{}^{+}\mathrm{F}_{\mu\nu}\,, \qquad {}^{+}\mathrm{W}_{\mu\nu} = -i\,\mathrm{W}_{\mu\nu}\,, \tag{76}$$

then, equations (75) are equivalent to a phase change of this tensor:

$$\mathrm{W}'_{\mu\nu} = e^{-i\xi}\,\mathrm{W}_{\mu\nu}\,, \tag{77}$$

similar to (73).

The electromagnetic field possesses two independent Lorentz invariants, namely:

$$\mathrm{I}_1 = \mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} = 2(c^2\mathrm{B}^2 - \mathrm{E}^2), \qquad \mathrm{I}_2 = {}^{+}\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} = 4c\,\vec{\mathrm{E}}\cdot\vec{\mathrm{B}}\,, \tag{78}$$

which can also be expressed via the Riemann-Silberstein complex vector, or by means of (76):

$$\vec{\mathrm{F}}\cdot\vec{\mathrm{F}} = \frac{1}{2}(\mathrm{I}_1 + i\,\mathrm{I}_2)\,, \qquad \mathrm{W}^{\mu\nu}\,\mathrm{W}_{\mu\nu} = 2(\mathrm{I}_1 + i\,\mathrm{I}_2)\,; \tag{79}$$

the Faraday tensor (67) is Null when $\mathrm{I}_1 = \mathrm{I}_2 = 0$. In each point of the spacetime we have the ensemble (52) which acts as a generator of any skew-symmetric tensor of second order, thus:

$$\mathrm{F}_{\mu\nu} = \sum_{r=1}^{6} p_r\,\mathrm{W}_{(r)\,\mu\nu}\,, \tag{80}$$

where the real quantities $p_j$ are the components of $\mathrm{F}_{\alpha\beta}$ in the basis (52), and these six scalars contain the same information as the components $\mathrm{E}_j$ and $\mathrm{B}_j$, $j = 1, 2, 3$ of the electric and magnetic fields, respectively. Using in (80) the relations (54) and (55) we obtain:

$${}^{+}\mathrm{F}_{\alpha\beta} = p_1\mathrm{W}_{(6)\,\alpha\beta} - p_2\mathrm{W}_{(5)\,\alpha\beta} + p_3\mathrm{W}_{(4)\,\alpha\beta} - p_4\mathrm{W}_{(3)\,\alpha\beta} + p_5\mathrm{W}_{(2)\,\alpha\beta} - p_6\mathrm{W}_{(1)\,\alpha\beta}\,, \tag{81}$$

which together with (52), (53), (78) and (80) permits to prove the important identities [50, 51, 53]:

$$\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\alpha\nu} - {}^{+}\mathrm{F}^{\mu\nu}\,{}^{+}\mathrm{F}_{\alpha\nu} = \frac{1}{2}\mathrm{I}_1\delta^{\mu}_{\alpha}\,, \qquad {}^{+}\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\alpha\nu} = \frac{1}{4}\mathrm{I}_2\delta^{\mu}_{\alpha}\,, \tag{82}$$

These relations play an important role in the calculation of the exponential of a skew-symmetric matrix [74], a relevant object to determine the motion of a point charge immerse in a uniform electromagnetic field [34, 37, 50, 52, 73, 75-79]; furthermore:

$$\mathrm{I}_1 = 2(p_4^2 + p_5^2 + p_6^2 - p_1^2 - p_2^2 - p_3^2), \qquad \mathrm{I}_2 = 4(p_1\, p_6 + p_3\, p_4 - p_2\, p_5)\,. \tag{83}$$

If we employ the spinors (56) in (80), an interesting relation is obtained (which verifies the statement (57)):

$$\mathrm{F}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = \Phi_{\mathrm{AC}}\,\epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}} + \epsilon_{\mathrm{AC}}\,\Phi_{\dot{\mathrm{B}}\dot{\mathrm{D}}}\,, \qquad \Phi_{\dot{\mathrm{B}}\dot{\mathrm{D}}} = \overline{\Phi_{\mathrm{BD}}}\,, \tag{84}$$

this shows the existence of the Maxwell's spinor:

$$\begin{aligned}
&\Phi_{\mathrm{AC}} = \Phi_{\mathrm{CA}} = \phi_0\, \iota_{\mathrm{A}}\iota_{\mathrm{C}} - \phi_1(o_{\mathrm{A}}\iota_{\mathrm{C}} + o_{\mathrm{C}}\iota_{\mathrm{A}}) + \phi_2\, o_{\mathrm{A}}o_{\mathrm{C}}\,,\\
&\phi_1 = \Phi_{\mathrm{AC}}\,\iota^{\mathrm{A}}o^{\mathrm{C}} = \tfrac{1}{2}(p_3 - i\,p_4)\,,\\
&\phi_0 = \Phi_{\mathrm{AC}}\,o^{\mathrm{A}}o^{\mathrm{C}} = \frac{1}{2}[-(p_1 + p_5) + i(p_2 + p_6)]\,,\\
&\phi_2 = \Phi_{\mathrm{AC}}\,\iota^{\mathrm{A}}\iota^{\mathrm{C}} = \frac{1}{2}[(p_1 - p_5) + i(p_2 - p_6)].
\end{aligned} \tag{85}$$

From (84) the relations (65) and (66) are immediate; moreover:

$$\Phi_{\mathrm{AC}} = -\tfrac{1}{2}\mathrm{F}_{\mathrm{AC}}{}^{\dot{\mathrm{B}}}{}_{\dot{\mathrm{B}}}\,, \qquad \Phi_{\dot{\mathrm{B}}\dot{\mathrm{D}}} = -\tfrac{1}{2}\mathrm{F}^{\mathrm{C}}{}_{\mathrm{C}\dot{\mathrm{B}}\dot{\mathrm{D}}}\,, \tag{86}$$

then it is clear that the symmetric 2-spinor encodes all the electromagnetic information, which is more remarkable when are used (66) and (84) to obtain the spinorial version of Faraday's complex spinor (76):

$$\mathrm{W}_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = 2\,\Phi_{\mathrm{AC}}\,\epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}}\,, \tag{87}$$

therefore:

$$\Phi^{\mathrm{AC}}\,\Phi_{\mathrm{AC}} = 2[\phi_0\phi_2 - (\phi_1)^2] = \tfrac{1}{8}\,\mathrm{W}^{\mu\nu}\mathrm{W}_{\mu\nu} = \tfrac{1}{4}(\mathrm{I}_1 + i\,\mathrm{I}_2), \tag{88}$$

due to (79). Thus it is evident that [80] $\Phi^{\mathrm{AC}}\Phi_{\mathrm{AC}} = 0 \;\Leftrightarrow\; \mathrm{I}_1 = \mathrm{I}_2 = 0$. If we substitute in (88) the expressions (85) for $\phi_0$, $\phi_1$ and $\phi_2$ we automatically recover (83).

In (80) the real tetrad generates the Faraday tensor, then it seems natural to wonder how this would be written in terms of the NP null tetrad, which can be solved employing (58) and (85) into (87) to deduce the tensorial counterpart:

$$\mathrm{W}_{\mu\nu} = \mathrm{F}_{\mu\nu} + i\,{}^{*}\mathrm{F}_{\mu\nu} = 2\big(\phi_0\,\mathrm{U}_{\mu\nu} + \phi_1\,\mathrm{M}_{\mu\nu} + \phi_2\,\mathrm{V}_{\mu\nu}\big)\,, \tag{89}$$

with the definitions (59), together with the properties (43) allow to determine the components:

$$\phi_0 = \mathrm{F}_{\mu\nu}\,l^{\mu}m^{\nu}\,, \qquad \phi_1 = \tfrac{1}{2}\mathrm{F}_{\mu\nu}(l^{\mu}n^{\nu} + \overline{m}^{\mu}m^{\nu})\,, \qquad \phi_2 = \mathrm{F}_{\mu\nu}\,\overline{m}^{\mu}n^{\nu}\,, \tag{90}$$

which are fundamental to study Maxwell's electrodynamics by means of the Newman-Penrose formalism [39, 41, 42, 45, 81, 82]. From (85), the corresponding components of $F_{\mu\nu}$ on the real tetrad are:

$$p_1 = \frac{1}{2}(\phi_2 - \phi_0) + \text{cc}\,, \qquad p_2 = \frac{i}{2}(\bar{\phi}_0 + \bar{\phi}_2) + \text{cc}\,, \qquad p_3 = \phi_1 + \bar{\phi}_1\,,$$
$$p_4 = i(\phi_1 - \bar{\phi}_1)\,, \quad p_5 = -\frac{1}{2}(\phi_0 + \phi_2) + \text{cc}\,, \quad p_6 = \frac{i}{2}(\bar{\phi}_0 - \bar{\phi}_2) + \text{cc}\,, \tag{91}$$

where cc denotes the complex conjugate of all the previous terms.

With the aid of (86) the components of Maxwell's spinor can be constructed:

$$\Phi_{11} = -F_{11\dot{2}\dot{1}}\,, \qquad \Phi_{12} = \frac{1}{2}(F_{12\dot{1}\dot{2}} - F_{12\dot{2}\dot{1}})\,, \qquad \Phi_{22} = -F_{22\dot{2}\dot{1}}\,, \tag{92}$$

in function of the components of the spinor (84), which in turn depend on the Faraday tensor:

$$F_{11\dot{2}\dot{1}} = \frac{1}{2}[(F_{10} + F_{13}) + i\,(F_{23} + F_{20})]\,, \quad F_{12\dot{1}\dot{2}} = F_{30}\,, \quad F_{12\dot{2}\dot{1}} = i\,F_{21}\,, \quad F_{0r} = E_r,\ r = 1, 2, 3$$
$$F_{22\dot{2}\dot{1}} = \frac{1}{2}[(-F_{10} + F_{13}) + i\,(-F_{23} + F_{20})]\,, \quad F_{32} = cB_1\,, \quad F_{13} = cB_2\,, \quad F_{21} = cB_3\,, \tag{93}$$

as a result we obtain the symmetric Maxwell spinor in terms of the Riemann-Silberstein complex vector indicated in (73):

$$(\Phi_{AB}) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -(F_2 + i\bar{F}_1) & iF_3 \\ iF_3 & -F_2 + iF_1 \end{pmatrix}, \tag{94}$$

thus:

$$F_1 = i\,(\Phi_{11} - \Phi_{22})\,, \qquad F_2 = -(\Phi_{11} + \Phi_{22})\,, \qquad F_3 = -2i\,\Phi_{12}\,, \tag{95}$$

and the components of the electric and magnetic fields are:

$$E_1 = \frac{1}{2}(\Phi_{11} - \Phi_{22}) + \text{cc}\,, \quad E_2 = -\frac{i}{2}(\Phi_{11} + \Phi_{22}) + \text{cc}\,, \quad E_3 = -(\Phi_{12} + \Phi_{\dot{1}\dot{2}})\,,$$
$$cB_1 = \frac{i}{2}(-\Phi_{11} + \Phi_{22}) + \text{cc}\,, \quad cB_2 = -\frac{1}{2}(\Phi_{11} + \Phi_{22}) + \text{cc}\,, \quad cB_3 = i\,(\Phi_{12} - \Phi_{\dot{1}\dot{2}})\,. \tag{96}$$

Under a duality rotation the transformation rules (73) and (77) are verified, therefore from (87) and (94) we get:

$$\Phi_{AB} \longrightarrow e^{i\xi}\,\Phi_{AB}\,, \qquad W_{AC\dot{B}\dot{D}} \longrightarrow e^{i\xi}\,W_{AC\dot{B}\dot{D}}\,. \tag{97}$$

As is well known [22, 83] the Maxwell equations in vacuum exhibited in (70) adopt the tensorial form $F^{\mu\nu}{}_{,\mu} = {}^{+}F^{\mu\nu}{}_{,\mu} = 0$, namely:

$$W^{\mu\nu}{}_{,\mu} = 0\,, \tag{98}$$

whose spinorial version can be obtained via the prescription (16) assisted by (87) [20, 39, 41, 42, 84]:

$$\partial_{B\dot{C}}\,\Phi^{AB} \;\equiv\; \Phi^{AB}{}_{,B\dot{C}} = 0\,, \qquad A, B = 1, 2\,, \tag{99}$$

where $\partial_{B\dot{C}} = \sigma^{\mu}{}_{B\dot{C}}\,\partial_{\mu}$ :

$$\partial_{1\dot{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial z}\right), \quad \partial_{1\dot{2}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \quad \partial_{2\dot{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

$$\partial_{2\dot{2}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{c\partial t} - \frac{\partial}{\partial z}\right), \tag{100}$$

notice that $\overline{\partial_{A\dot{B}}} = \partial_{B\dot{A}}$ .

Lanczos [85] was the first to write the Maxwell equations (70) in quaternionic form [86-89]:

$$\mathbf{\nabla}\,\mathbf{F} = \mathbf{0}\,, \tag{101}$$

where:

$$\mathbf{\nabla} = \frac{i}{c}\frac{\partial}{\partial t} + \mathbf{I}\,\frac{\partial}{\partial x} + \mathbf{J}\,\frac{\partial}{\partial y} + \mathbf{K}\,\frac{\partial}{\partial z}\,, \quad \mathbf{F} = F_1\,\mathbf{I} + F_2\,\mathbf{J} + F_3\,\mathbf{K}\,,$$

$$F_r = cB_r + i\,E_r, \quad r = 1, 2, 3 \tag{102}$$

If we recall that every quaternion is isomorphic to a complex matrix 2x2, see (6), then the matrix version of (101) turns out to be (99), consult [90].

With the material exposed in this Section it can be undertaken the study of the eigenvalue problem of $F_{\mu\nu}$ and $\Phi_{AB}$ , however, we do not pursue this task here because it has already been addressed elsewhere [53, 91].

Now, let us consider the energy-momentum tensor of the electromagnetic field $T^{\mu\nu} = T^{\nu\mu}$ whose symmetry was required by Planck [92] to secure the relativistic equivalence between mass and energy, furthermore, it has null trace ($T^{\mu}{}_{\mu} = 0$) because the photon being massless. The photon has spin 1 because the Maxwell spinor possesses two indices [29]. Thus, it exists the following quadratic expression in the Faraday tensor [22]:

$$T^{\mu\nu} = -F^{\mu\alpha}\,F^{\nu}{}_{\alpha} + \tfrac{1}{4}\,I_1\,g^{\mu\nu}\,, \qquad I_1 = F^{\alpha\beta}F_{\alpha\beta}\,, \tag{103}$$

which is invariant under duality rotations as expected [that is, (103) is not altered if we use the fields (75)], due to the fact that this symmetry is associated with the conservation of the electromagnetic energy, see (74).

With the aid of relations (82) it is straightforward to prove the famous Rainich identity [54, 56, 93-95]:

$$T^{\mu\alpha}\, T_{\nu\alpha} = \frac{1}{4}\left(T^{\alpha\beta} T_{\alpha\beta}\right) \delta^{\mu}_{\nu} = \frac{1}{16}\left(I_1^2 + I_2^2\right) \delta^{\mu}_{\nu}\,, \qquad I_2 = {}^{+}F^{\alpha\beta} F_{\alpha\beta}\,. \tag{104}$$

Besides, equations (82) permit to write (103) in terms of the dual tensor (54):

$$T^{\mu\nu} = -\frac{1}{2}\left(F^{\mu\alpha}\, F^{\nu}{}_{\alpha} + {}^{+}F^{\mu\alpha}\, {}^{+}F^{\nu}{}_{\alpha}\right), \tag{105}$$

where the spinors (66) and (84) can be used to deduce the spinor associated with the Maxwell tensor:

$$T_{AC\dot{B}\dot{D}} = 2\, \Phi_{AC}\, \Phi_{\dot{B}\dot{D}}\,. \tag{106}$$

## 5. CONCLUSIONS

The present work was dedicated to the algebraic subtleties of the spinorial analysis; in a forthcoming paper we will develop the differential aspects with the aim to apply them to the Lanczos potential [96-98]. Let us conclude quoting Sir Michael Atiyah in an e-mail sent to Farmelo [99] the 15th July, 2007:

> "No one fully understands spinors. Their algebra is formally understood but their geometrical significance is mysterious. In some sense they describe the 'square-root' of geometry and, just as understanding the concept of the square root of -1 took centuries, the same might be true of spinors".

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