Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 94(2) (2018) 72-98 EISSN 2392-2192

Mellin transform in higher dimensions for the valuation of the European basket put option with multi-dividend paying stocks

Sunday Emmanuel Fadugba

Department of Mathematics, Ekiti State University, Ado Ekiti, Nigeria

E-mail address: sunday.fadugba@eksu.edu.ng

ABSTRACT

Numerical approximations and analytical techniques have been proposed for the pricing of basket put option but there is no known integral equation for the valuation of European basket put option with multi-dividend yields. Mellin transform is useful when dealing with the unstable mathematical system. This paper presents the integral equation for the price of the European basket put option which pays multi-dividend yields by means of the Mellin transform in higher dimensions that enables option equations to be solved directly in terms of market prices rather than log-prices, providing a more natural setting to the problem of pricing. The expression for the integral equation for the valuation of the European basket put option was obtained by solving the multi-dimensional partial differential equation for the price of the option via the multi-dimensional Mellin transform. The analytical solution to the derived integral equation for the case of two-dividend paying stocks was obtained. Also the effect of the correlation coefficients on the price of the European basket put option was considered. A comparative study of the Mellin transform, Monte Carlo method and implied binomial model for the valuation of the option in the case of was considered. The numerical results showed that negatively correlated assets are more sensitive to correlation changes than positively correlated assets as shown in Tables 1 and 2**.** Also the numerical evaluation of our expression is more efficient and produces a comparable result than the other methods. Hence the Mellin transform is a good approach for the valuation of European basket put option with multi-dividend yields.

Keywords**:** European basket option, Dividend paying stock, Generalized Black-Scholes equation, Mellin transform

Mathematics Subject Classification: 35R60, 60H05, 60H10, 60H35, 91G20

(Received 17 January 2018; Accepted 07 February 2018; Date of Publication 08 February 2018)

1. INTRODUCTION

Nowadays, investment companies use financial derivatives for their risk management through hedging against possible fluctuations of the underlying asset price. Hence the valuation of financial derivative is an important field of financial research. An option is a financial derivative that grants its holder the right without obligation to buy (sell) a specific asset on or before a given date in the future for an agreed price called the exercise price. There are two basic types of options, namely; a call option which gives its holder the right to buy the underlying asset. A put option that gives its holder the right to sell an underlying asset.

A multi-asset option is path dependent option that depends on more than one underlying asset with the payoff defined by a function of the asset prices. An example of a multi-asset option is a basket option on *n* assets. A basket option is a financial contract with a portfolio of several underlying equity assets.

Basket options are becoming increasingly widespread in commodity and particularly energy markets. The volatility of the basket is lower than the individual volatilities of the stocks and therefore these options are popular as hedging tools. The valuation of basket options is a challenging task because the underlying value is a weighted sum of individual asset prices.

A basket option gives the holder the right, but not the obligation, to buy or sell a group of underlying assets. The payoff for a basket call option is given by

$$\text{Payoff(basket call)} = \max\left(\sum_{j=1}^{n}\alpha_j S_j - K, 0\right) \tag{1}$$

The payoff for a basket put option is of the form:

$$\text{Payoff(basket put)} = \max\left(K - \sum_{j=1}^{n}\alpha_j S_j, 0\right) \tag{2}$$

where α_j is the number of shares of asset *j* in the basket, S_j is the price of asset *i* in the basket, *K* is the exercise price and $\sum_{j=1}^{n}\alpha_j = 1$.

The revolution on financial derivative both in exchange markets and in academic communities began in the early 1970's. How to rationally price an option was not clear until 1973, when Black and Scholes [3] published their seminal work on option pricing in which they described a mathematical framework for finding the fair price of a European option. They used a no-arbitrage argument to describe a partial differential equation which governs the evolution of the option price with respect to the maturity time and underlying asset price.

Option price estimation as the most interesting of the derivative has many approaches and diverse qualities [8]. The subject of numerical methods in the area of options valuation and hedging is very broad. A wide range of different types of contracts is available and in many cases there are several candidate models for the stochastic evolution of the underlying state variables.

Fadugba and Nwozo [4] used an integral method based on the double Mellin transform to derive the integral representations for the price of European and American put options on a basket of two-dividend paying stocks. They deduced that by the decomposition of the price of the American put option on a basket of two stocks, its counterpart "European put option" can be obtained directly. Basket option pricing using Mellin transforms was considered by Manuge and Kim [7]. They used the Mellin transform to derive the analytical pricing formulas and Greeks for American basket put options. They assumed assets are driven by geometric Brownian motion which exhibit correlation and pay a continuous dividend rate. Panini and Srivastav [10] considered option pricing with Mellin transforms. They derived the integral representations for the price of European and American basket put options with non-dividend yield using the Mellin transform techniques.

For mathematical background, applications of the Mellin transforms and various numerical methods for the valuation of basket options, see [1,2,5,6,9,12-14], just to mention a few.

In this paper, the Mellin transform in higher dimensions and its applications in the theory of a European put option on a basket of *n*-dividend paying stocks are presented. The rest of the paper is organized as follows. Section Two discusses the multi-dimensional Mellin transform. Section Three presents the derivation of the generalized Black-Scholes partial differential equation for the price of the European basket put option with multi-dividend paying stocks. In Section Four, the use of the Mellin transform in higher dimensions to derive the expression for the integral equation for the valuation of the option was considered. An analytic solution for the price of the European basket put option for the case of $n = 2$ that permits the use of numerical integration in two dimensions was derived. The European basket put option Greeks was also obtained. In Section Five, numerical experiments, discussion of results and conclusion were considered.

2. MELLIN TRANSFORM IN HIGHER DIMENSIONS

A natural extension of the Mellin transform exists for higher dimensions. The double Mellin transform was first introduced by [11]. He proved conditions for which the Mellin transform and its inverse exist.

Definition 1: Let $X = (x_1, x_2, ..., x_n)'$ and $\xi^* = (\xi_1, \xi_2, ..., \xi_n)'$. For a function $f(X) \in \mathrm{R}^{n+}$, the Multi-dimensional Mellin transform or (Mellin transform in higher dimensions) is a complex function

$$M(f(X);\xi^*) = \widehat{f}(\xi^*) = \int_{R^{n+}} f(\xi^*) X^{\xi^*-1} dX \tag{3}$$

Definition 2: Let $X = (x_1, x_2, ..., x_n)'$, $\xi^* = (\xi_1, \xi_2, ..., \xi_n)'$ and $f(\xi^*) \in \mathrm{C}^n$ be analytic on $\gamma = \times_{j=1}^{n} \gamma_j$ where $\gamma_j = \{c_j + ib_j : c_j \in R, b_j = \pm\infty\}$ are the strips in C^n. The inverse multidimensional Mellin transform is a continuous function $f(X) \in \mathrm{R}^{n+}$ of the form

$$M^{-1}(\hat{f}(\xi^*);X) = f(X) = \frac{1}{(2\pi i)^n}\int_\gamma \hat{f}(\xi^*)X^{-\xi^*}d\xi^* \tag{4}$$

Remark 2.1

(i) If $f(X) = \prod_{j=1}^{n} f_j(x_j)$, then it follows that $\hat{f}(\xi^*) = \prod_{j=1}^{n} M\left(f_j(x_j);\xi_j\right)$

(ii) From (i), it is clearly seen that the properties of the univariate Mellin transform can be used to obtain solutions of multi-dimensional Mellin transforms.

The extension of the existence theorem of the univariate Mellin transform [5] to the multidimensional Mellin transform is given below.

Theorem 2.1 (Existence Theorem for Multidimensional Mellin Transform)

Let $f(X) \in \mathrm{R}^{n+}$ *be a* locally integrable *function*[1] *such that*

$$f(X) = \begin{cases} O(X^{a_j}),\ X \to 0 \\ O(X^{d_j}),\ X \to \infty \end{cases} \tag{5}$$

Then the multidimensional Mellin transform $\hat{f}(\xi^*)$ exists for any $\xi^* \in \mathrm{C}^n$ on $-a_j < \Re(\xi^*) < -d_j$

Remark 2.2

The interval (a_j, d_j) is known as fundamental strip (domain of analyticity) for $\hat{f}(\xi^*)$. The existence is granted for locally integrable function $f(X) \in \mathrm{R}^{n+}$ since the exponent in the order at 0 is strictly greater than the exponent of the order at ∞.

The following derivative properties of the multi-dimensional Mellin transform are useful in this paper.

Let $f(X) \in \mathrm{R}^{n+}$ be twice differentiable with respect to x_j and x_k. If $\prod_{j=1}^{n} x_j^{\xi_j} f(X)$ vanishes as $x_j \to 0$ and $x_j \to \infty$ then,

$$M\left(x_j x_k \frac{d^2}{dx_j x_k} f(X);\xi^*\right) = \xi_j(\xi_j - 1)\hat{f}(\xi^*),\quad j = k \tag{6}$$

$$M\left(x_j x_k \frac{d^2}{dx_j x_k} f(X);\xi\right) = \xi_j(\xi_k)\hat{f}(\xi)\qquad j \neq k \tag{7}$$

---

[1] A function *f(x)* is said to be locally integrable, if around every point in the domain, there is a neighborhood on which the function is integrable. Examples are integrable functions and constant continuous function on R e.g. *f(x)* =1 on R

3. THE GENERALIZED BLACK-SCHOLES-MERTON PARTIAL DIFFERENTIAL EQUATION WITH MULTI-DIVIDEND PAYING STOCKS

For an option on n stocks, it is assumed that the stock prices $S_j,\ j=1,2,3,...,n$ follow correlated geometric Brownian motion with drift $\mu_j,\ j=1,2,3,...,n$ and volatility $\sigma_j,\ j=1,2,3,...,n.$

Therefore

$$dS_j = \mu_j S_j dt + \sigma_j S_j dW_{t_j} \tag{8}$$

where each W_{t_j} is a standard Brownian motion and dW_{t_j} are normally distributed random variables with mean zero, variance dt and $\rho_{jk} = corr(dW_{t_j}, dW_{t_k}) \in [-1,1],$ for $j \neq k$. Before the derivation of the generalized Black-Scholes-Merton partial differential equation, a counterpart; the It $\hat{o}$'s lemma in higher dimensions is needed. Consider a multi-dimensional It$\hat{o}$ process of the form

$$dX_{tj} = \mu_j(X_{tj}, t)dt + \sigma_j(X_{tj}, t)dW_{tj}, \quad 1 \le j \le n \tag{9}$$

Therefore, the multi-dimensional It $\hat{o}$'s lemma for (9) is given by

$$dv(X_t,t) = \left(\frac{\partial v(X_t,t)}{\partial t} + \sum_{j=1}^{n} \mu_j \frac{\partial v(X_t,t)}{\partial x_j} + \frac{1}{2} \sum_{j,k=1}^{n} \rho_{jk} \sigma_j \sigma_k x_j x_k \frac{\partial^2 v(X_t,t)}{\partial x_j \partial x_k} \right) dt + \sum_{j=1}^{n} \sigma_j \frac{\partial v(X_t,t)}{\partial x_j} dW_j \tag{10}$$

where $v(X_t,t) \in C^{2,1}$.

The derivation of the generalized Black-Scholes partial differential equation for the European basket put option with multi-dividend paying stocks is given in the following result.

Theorem 3.1

Let $E_p(S_1,..S_n,t) \in \mathrm{C}^{2,1}$ be some functions with multi-dividend paying stocks, S_j be the current price of the underlying asset j, S_k be the current price of the underlying asset k, K be the strike price, σ_j be the volatility of asset j, σ_k be the volatility of asset k, ρ_{jk} be the correlation coefficient between asset j and asset k, T be the time to expiry, r be the risk-free interest rate and q_i be the multi-dividend yields. Using the Itô's lemma (10), the generalized Black-Scholes partial differential equation for the price of an European basket put option is obtained as

$$\frac{\partial E_p(S_1,...,S_n,t)}{\partial t}+\sum_{j=1}^{n}(r-q_j)S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}+\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k S_j S_k\frac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k} - rE_p(S_1,...,S_n,t)=0 \quad (11)$$

Proof: Construct a portfolio consisting of an option $E_p(S_1,...,S_n,t)$ and $\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}$ amount of assets S_j. Let the asset S_j be driven by correlated geometric Brownian motions. Using (10) for some functions $E_p(S_1,...,S_n,t)$ with multi- dividend paying stocks, yields

$$dE_p(S_1,...,S_n,t)=\frac{\partial E_p(S_1,...,S_n,t)}{\partial t}dt+\sum_{j=1}^{n}(\mu_j-q_j)S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}dt + \frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k S_j S_k\frac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}dt+\sum_{j=1}^{n}\sigma_j S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}dW_{t_j} \quad (12)$$

where q_j is the multi-dividend yield. The value of the portfolio denoted by θ is given by

$$\theta=E_p(S_1,...,S_n,t)-\sum_{j=1}^{n}S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j} \quad (13)$$

After one time step *dt*, the value of the portfolio changes by

$$d\theta=dE_p(S_1,...,S_n,t)-\sum_{j=1}^{n}\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}dS_j \quad (14)$$

Substituting (8) and (12) into (14) yields

$$d\theta=\left(\frac{\partial E_p(S_1,...,S_n,t)}{\partial t}-\sum_{j=1}^{n}q_jS_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}+\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k S_j S_k\frac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}\right)dt \quad (15)$$

The portfolio is now riskless due to the elimination of dW_{t_j} term. It must then earn a return similar to other short term riskless securities such as bank account. Therefore

$$d\theta=r\theta dt \quad (16)$$

where *r* is the riskless interest rate. Substituting (13) and (15) into (16) gives

$$\left(\frac{\partial E_p(S_1,...,S_n,t)}{\partial t}-\sum_{j=1}^{n}q_jS_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}+\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_kS_jS_k\frac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}\right)dt$$
$$=r\left(E_p(S_1,...,S_n,t)-\sum_{j=1}^{n}S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}\right)dt$$

Rearranging and solving the last equation further, the multi-dimensional Black-Scholes partial differential equation for the price of an option $E_p(S_1,...,S_n,t)$ with multi-dividend paying stocks is obtained as

$$\frac{\partial E_p(S_1,...,S_n,t)}{\partial t}+\sum_{j=1}^{n}(r-q_j)S_j\frac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}$$
$$+\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_kS_jS_k\frac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}-rE_p(S_1,...,S_n,t)=0$$

Hence (11) is established.

Remark 3.1

i. Equation (11) is the generalized Black-Scholes-Merton partial differential equation for the price of European basket put option.

ii. Setting $n=1$, (11) becomes the well-known Black-Scholes-Merton partial differential equation for the price of plain European put option given by

$$\frac{\partial E_p(S,t)}{\partial t}+rS\frac{\partial E_p(S,t)}{\partial S}+\frac{\sigma^2S^2}{2}\frac{\partial^2 E_p(S,t)}{\partial S^2}-rE_p(S,t)=0$$

iii. Setting $E_p(S_1,...,S_n,t)=P_E(S_1,...,S_n,t),\ q_j(j=1,2,...,n)=0$, the generalized Black-Scholes partial differential equation for the price of European basket put option $P_E(S_1,...,S_n,t)$ with non-dividend paying stocks is obtained as

$$\frac{\partial P_E(S_1,...,S_n,t)}{\partial t}+\sum_{j=1}^{n}rS_j\frac{\partial P_E(S_1,...,S_n,t)}{\partial S_j}$$
$$+\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_kS_jS_k\frac{\partial^2 P_E(S_1,...,S_n,t)}{\partial S_j\partial S_k}-rP_E(S_1,...,S_n,t)=0 \quad (17)$$

4. MULTI-DIMENSIONAL MELLIN TRANSFORM FOR THE VALUATION OF THE EUROPEAN PUT OPTION ON A BASKET OF MULTI-DIVIDEND PAYING STOCKS

The following result gives the derivation of the expression for the integral equation for the price of the European basket put option by means of the Mellin transform in higher dimensions.

Theorem 4.1

The generalized Black-Scholes partial differential equation for the price of the European basket put option is given by

$$\left.\begin{array}{l}\dfrac{\partial E_p(S_1,...,S_n,t)}{\partial t}+\sum_{j=1}^{n}(r-q_j)S_j\dfrac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}+\dfrac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k S_j S_k\dfrac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}\\ \qquad -rE_p(S_1,...,S_n,t)=0, \qquad\qquad 0<S_1,...S_n<\infty, 0\le t\le T\end{array}\right\} \tag{18}$$

with the initial and boundary conditions

$$\left.\begin{array}{ll} E_p(S_1,...S_n,t)=\phi(S_1,...,S_n)=\left(K-\sum_{j=1}^{n}S_j\right)^{+} & \text{on } [0,\infty) \\ \lim_{S_j\to 0}E_p(S_1,...S_n,t)=Ke^{-r(T-t)} & \text{on } [0,T) \\ \lim_{\sum_{i=1}^{n}S_j\to\infty}E_p(S_1,...S_n,t)=0 & \text{on } [0,T) \end{array}\right\} \tag{19}$$

Then, the expression for the integral equation for the price of the European put option on a basket of multi-dividend paying stocks is obtained as

$$\left.\begin{array}{l} E_p(S_1,...,S_n,t)=(2\pi i)^{-n}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}...\int_{c_n-i\infty}^{c_n+i\infty}\left(\dfrac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{i=1}^{n}\xi_i\right)}}{(\xi_1+\xi_2+...+\xi_n)(\xi_1+...+\xi_n+1)}\right)\times \\ \qquad \left(e^{G(\xi_1,...,\xi_n)(T-t)}S_1^{-\xi_1}...S_n^{-\xi_n}\right)d\xi_1...d\xi_n \end{array}\right\} \tag{20}$$

Proof: It is assumed that the underlying assets pay multi-dividend yields and follow geometric Brownian motion. From (18);

$$\left.\begin{array}{l}\dfrac{\partial E_p(S_1,...,S_n,t)}{\partial t}+\sum_{j=1}^{n}(r-q_j)S_j\dfrac{\partial E_p(S_1,...,S_n,t)}{\partial S_j}+\dfrac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k S_j S_k\dfrac{\partial^2 E_p(S_1,...,S_n,t)}{\partial S_j\partial S_k}\\ \qquad -rE_p(S_1,...,S_n,t)=0, \qquad\qquad 0<S_1,...S_n<\infty, 0\le t\le T\end{array}\right\}$$

To use the multi-dimensional Mellin transform, assume that $E_p(S_1,...,S_n,t)$ satisfies the condition

$$E_p(S_1,...,S_n,t)=\begin{cases} O(1) & \text{for } S_1,...,S_n \to 0^+ \\ O(S_1,...,S_n) & \text{for } S_1,...,S_n \to \infty \end{cases} \tag{21}$$

Equation (21) guarantees the existence of the multi-dimensional Mellin transform. Let $\widehat{E}_p(\xi_1,...,\xi_n,t)$ denote the multi-dimensional Mellin transform of $E_p(S_1,...,S_n,t)$ given by

$$\widehat{E}_p(\xi_1,...,\xi_n,t)=\int_0^\infty\int_0^\infty...\int_0^\infty E_p(S_1,...,S_n,t)S_1^{\xi_1-1}...S_n^{\xi_n-1}dS_1...dS_n \tag{22}$$

where the complex variable ξ^* exists in an appropriate domain of convergence in C^n. The functions $E_p(S_1,...,S_n,t)$ and $\widehat{E}_p(\xi_1,...,\xi_n,t)$ are called a Mellin transform pair. Conversely, the multi-dimensional Mellin transform inversion formula for the European put option $E_p(\xi_1,...,\xi_n,t)$ on a basket of multi-dividend paying stocks is given by

$$E_p(S_1,...,S_n,t)=(2\pi i)^{-n}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}...\int_{c_n-i\infty}^{c_n+i\infty}\widehat{E}_p(\xi_1,...,\xi_n,t)S_1^{-\xi_1}...S_n^{-\xi_n}d\xi_1...d\xi_n \tag{23}$$

with $c_j \in \Re(\xi^*),(j=1,2,...,n)$ and $\Re(\xi^*)>0$.

Thus to find the multi-dimensional Mellin transform of (18), applying the definition given by (22) to (18), yields

$$\frac{d\widehat{E}_p(\xi_1,...,\xi_n,t)}{dt}+\left(\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k\xi_j\xi_k+\frac{1}{2}\sum_{j=1}^{n}\sigma_j^2\xi_j-\sum_{j=1}^{n}(r-q_j)\xi_j-r\right)\widehat{E}_p(\xi_1,...,\xi_n,t)=0 \tag{24}$$

Equation (24) becomes

$$\frac{d\widehat{E}_p(\xi_1,...,\xi_n,t)}{dt}+G(\xi_1,...,\xi_n)\widehat{E}_p(\xi_1,...,\xi_n,t)=0 \tag{25}$$

where

$$G(\xi_1,...,\xi_n)=\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k\xi_j\xi_k-\sum_{j=1}^{n}\left((r-q_j)-\frac{\sigma_j^2}{2}\right)\xi_j-r \tag{26}$$

The solution to (25) is obtained as

$$\widehat{E}_p(\xi_1,...,\xi_n,t) = H(\xi_1,...,\xi_n)e^{-G(\xi_1,...,\xi_n)t} \tag{27}$$

where $H(\xi_1,...,\xi_n)$ is the constant of integration given by

$$H(\xi_1,...,\xi_n) = \widehat{\phi}(\xi_1,...,\xi_n)e^{G(\xi_1,...,\xi_n)T} \tag{28}$$

The multi-dimensional Mellin transform of the final condition $\phi(S_1,...,S_n) = \left(K - \sum_{j=1}^{n} S_j\right)^+$ in (19) is obtained as

$$\widehat{\phi}(\xi_1,...,\xi_n) = \frac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{j=1}^{n}\xi_j\right)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)} \tag{29}$$

where

$$B(\xi_1,...,\xi_n) = \frac{\prod_{j=1}^{n}\Gamma(\xi_j)}{\Gamma\left(\sum_{j=1}^{n}\xi_j\right)} \tag{30}$$

Equation (30) is called the multinomial beta function of *n*-variables.
Substituting (29) into (28) yields

$$H(\xi_1,...,\xi_n) = \frac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{j=1}^{n}\xi_j\right)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)}\exp(G(\xi_1,...,\xi_n)T) \tag{31}$$

Therefore, (27) becomes

$$\widehat{E}_p(\xi_1,...,\xi_n,t) = \frac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{j=1}^{n}\xi_j\right)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)}e^{G(\xi_1,...,\xi_n)(T-t)} \tag{32}$$

Taking the inverse multi-dimensional Mellin transform of (32), the expression for the integral equation for the price of the European put option on a basket of multi-dividend paying stocks is obtained as

$$E_p(S_1,...,S_n,t)=(2\pi i)^{-n}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}...\int_{c_n-i\infty}^{c_n+i\infty}\left(\frac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{i=1}^{n}\xi_j\right)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)}\right)\times \left(e^{G(\xi_1,...,\xi_n)(T-t)}S_1^{-\xi_1}...S_n^{-\xi_n}\right)d\xi_1...d\xi_n$$

Hence (20) is established.

Remark 4.1

(i) For $n=1$, (20) becomes the univariate Mellin-type formula for a plain European put option given by

$$E_p(S,t)=(2\pi i)^{-1}\int_{c_1-i\infty}^{c_1+i\infty}\left(\frac{K^{(1+\xi)}}{\xi(\xi+1)}\exp(G(\xi)(T-t))S^{-\xi}\right)d\xi \tag{33}$$

With

$$G(\xi)=\frac{1}{2}\sigma^2\xi^2-\left((r-q)-\frac{\sigma^2}{2}\right)\xi-r \tag{34}$$

(ii) For $n=2$, (20) becomes the integral equation for the price of the European put option on a basket of two-dividend paying stocks via the double Mellin transform given by

$$\left.\begin{aligned}&E_p(S_1,S_2,t)=(2\pi i)^{-2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\frac{B(\xi_1,\xi_2)K^{\left(1+\sum_{j=1}^{2}\xi_j\right)}}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)}e^{G(\xi_1,\xi_2)(T-t)}S_1^{-\xi_1}S_2^{-\xi_2}d\xi_1d\xi_2\\&B(\xi_1,\xi_2)=\frac{\prod_{j=1}^{2}\Gamma(\xi_j)}{\Gamma\left(\sum_{j=1}^{2}\xi_j\right)},G(\xi_1,\xi_2)=\frac{1}{2}\sum_{j,k=1}^{2}\rho_{jk}\sigma_j\sigma_k\xi_j\xi_k-\sum_{j=1}^{2}\left((r-q_j)-\frac{\sigma_j^2}{2}\right)\xi_j-r\end{aligned}\right\} \tag{35}$$

(iii) For $n=3$, (20) becomes the triple Mellin-type formula for the price of the European put option on a basket of three-dividend paying stocks given by

$$E_p(S_1,S_2,S_3,t)=(2\pi i)^{-3}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_3-i\infty}^{c_3+i\infty}\left.\begin{cases}\left(\dfrac{B(\xi_1,\xi_2,\xi_3)K^{\left(1+\sum_{j=1}^{3}\xi_j\right)}}{(\xi_1+\xi_2+\xi_3)(\xi_1+\xi_2+\xi_3+1)}\right)\times \\ \left(\exp(G(\xi_1,\xi_2,\xi_3)(T-t))S_1^{-\xi_1}S_2^{-\xi_2}S_3^{-\xi_3}\right)d\xi_1 d\xi_2 d\xi_3\end{cases}\right\} \\ B(\xi_1,\xi_2,\xi_3)=\frac{\prod_{j=1}^{3}\Gamma(\xi_j)}{\Gamma\left(\sum_{j=1}^{3}\xi_j\right)},G(\xi_1,\xi_2,\xi_3)=\frac{1}{2}\sum_{j,k=1}^{3}\rho_{jk}\sigma_j\sigma_k\xi_j\xi_k-\sum_{j=1}^{3}\left((r-q_j)-\frac{\sigma_j^2}{2}\right)\xi_j-r \tag{36}$$

The multi-dimensional Mellin transform of the payoff function for the European basket put option is presented in the following result.

Theorem 4.2

Let the complex variable $\xi^*=(\xi_1,...,\xi_n)'$ exists in an appropriate domain of convergence in C^n, S_j be the current price of the underlying asset j, $0\le t\le T$ and $0<K,T,S_1,....,S_n<\infty$ for all $1\le j\le n$. For $\Re(\xi^*)>0$, then the multi-dimensional Mellin transform of the payoff function for the European basket put option is given by

$$\widehat{\phi}(\xi_1,...,\xi_n)=\frac{B(\xi_1,...,\xi_n)K^{(1+\xi_1+...+\xi_n)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)} \tag{37}$$

with

$$B(\xi_1,\xi_2)=\frac{\prod_{j=1}^{2}\Gamma(\xi_j)}{\Gamma\left(\sum_{j=1}^{2}\xi_j\right)}$$

Proof: Let the multi-dimensional Mellin transform of the European basket put payoff function be defined as

$$\widehat{\phi}(\xi_1,...,\xi_n)=\int_0^\infty\int_0^\infty...\int_0^\infty\phi(S_1,...,S_n)S_1^{\xi_1-1}...S_n^{\xi_n-1}dS_1...dS_n \tag{38}$$

Substituting the payoff function of the European basket option given by $\phi(S_1,...,S_n)=\left(K-\sum_{j=1}^{n}S_j\right)^+$ into (43), yields

$$M(\phi(S_1,...,S_n))=\widehat{\phi}(\xi_1,...,\xi_n)=\int_0^\infty\int_0^\infty...\int_0^\infty\left(K-\sum_{j=1}^{n}S_j\right)^+ S_1^{\xi_1-1}...S_n^{\xi_n-1}dS_1...dS_n \quad (39)$$

Solving (39) further leads to a relation

$$\widehat{\phi}(\xi_1,...,\xi_n)=\int_0^K\int_0^{K-S_1}...\int_0^{K-(S_1+...+S_{n-1})}\left(K-(S_1+...+S_n)\right)S_1^{\xi_1-1}...S_n^{\xi_n-1}dS_1...dS_n$$

$$=\int_0^K\int_0^{K-S_1}...\int_0^{K-(S_1+...+S_{n-1})}\left(K-(S_1+...+S_{n-1})-S_n\right)S_n^{\xi_n-1}S_1^{\xi_1-1}...S_{n-1}^{\xi_{n-1}-1}dS_1...dS_{n-1}$$

$$=\frac{\Gamma(\xi_n)}{\Gamma(\xi_n+2)}\int_0^K\int_0^{K-S_1}...\int_0^{K-(S_1+...+S_{n-2})}\left(K-(S_1+...+S_{n-2})-S_{n-1}\right)^{\xi_n+1}S_{n-1}^{\xi_{n-1}-1}S_1^{\xi_1-1}...S_{n-2}^{\xi_{n-2}-1}dS_1...dS_{n-2}$$

Simplifying the last expression further, the multi-dimensional Mellin transform of the payoff function for the European basket put option is obtained as

$$\widehat{\phi}(\xi_1,...,\xi_n)=\frac{\Gamma(\xi_n)\Gamma(\xi_{n-1})\Gamma(\xi_{n-2})...\Gamma(\xi_1)}{\Gamma(\xi_n+\xi_{n-1}+\xi_{n-2}+...+\xi_1+2)}K^{\xi_n+\xi_{n-1}+\xi_{n-2}+...+\xi_1+1}$$

$$=\frac{\prod_{j=1}^{n}\Gamma(\xi_j)}{\Gamma\left(2+\sum_{j=1}^{n}\xi_j\right)}K^{\left(1+\sum_{j=1}^{n}\xi_j\right)} \quad (40)$$

Hence (37) is established.

Remark 4.2

i. The above result depends on the final time condition for the price of European basket put option with multi-dividend paying stocks.

ii. For $n=1$, (37) leads to a relation $\widehat{\phi}(\xi)=\dfrac{K^{(1+\xi)}}{\xi(\xi+1)}$

4. 1. A closed form solution to the prece of European basket put option for the case of two-dividend paying stocks

This section presents a closed form solution for the valuation of the European basket put option with two-dividend paying stocks by means of the Mellin transform in higher dimensions.

Theorem 4.3

Let S_1 and S_2 be two underlying asset prices on dividend yields q_1 and q_2 respectively, $t \in [0,T]$ and $0 < K, T, r < \infty$. Then, the numerical solution to the integral equation for the price of the European put option on a basket of two-dividend paying stocks is given by

$$E_p(S_1,S_2,t) = \frac{Ke^{(-r(T-t)+C^T AC+B^T C)}}{4\pi^2 (\det(\mathrm{A}))^{0.5}} \sum_{j=1}^{M}\sum_{k=1}^{M} \xi_j \xi_k \Lambda(z_j,z_k) \tag{41}$$

where ξ_j's are the weights of the Gauss-Hermite quadrature method,

$$B^T = \left(\ln K - \ln S_1 - \left((r-q_1) - \frac{1}{2}\sigma_1^2 \right)(T-t) \quad \ln K - \ln S_2 - \left((r-q_2) - \frac{1}{2}\sigma_2^2 \right)(T-t) \right)$$

(42) $$A = \begin{pmatrix} \frac{1}{2}\sigma_1^2(T-t) & \frac{\rho\sigma_1\sigma_2(T-t)}{2} \\ \frac{\rho\sigma_1\sigma_2(T-t)}{2} & \frac{1}{2}\sigma_2^2(T-t) \end{pmatrix}, \; C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ and } C^T = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \tag{43}$$

Proof: From (35);

$$E_p(S_1,S_2,t) = (2\pi i)^{-2} \int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} \frac{B(\xi_1,\xi_2)K^{(1+\xi_1+\xi_2)}}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)} \exp(G(\xi_1,\xi_2)(T-t)) S_1^{-\xi_1} S_2^{-\xi_2} d\xi_1 d\xi_2 \tag{44}$$

where

$$B(\xi_1,\xi_2) = \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\xi_1+\xi_2)} \tag{45}$$

and

$$G(\xi_1,\xi_2) = \frac{1}{2}\sigma_1^2\xi_1^2 + \rho\sigma_1\sigma_2\xi_1\xi_2 + \frac{1}{2}\sigma_2^2\xi_2^2 - \left((r-q_1) - \frac{1}{2}\sigma_1^2 \right)\xi_1 - \left((r-q_2) - \frac{1}{2}\sigma_2^2 \right)\xi_2 - r \tag{46}$$

Setting

$$\left. \begin{aligned} &\gamma_1 = \frac{1}{2}\sigma_1^2(T-t), \gamma_2 = \rho\sigma_1\sigma_2(T-t), \gamma_3 = \frac{1}{2}\sigma_2^2(T-t), \\ &\gamma_4 = -\left((r-q_1) - \frac{1}{2}\sigma_1^2 \right)(T-t), \gamma_5 = \left((r-q_2) - \frac{1}{2}\sigma_2^2 \right)(T-t) \end{aligned} \right\} \tag{47}$$

Then,

$$G(\xi_1,\xi_2)(T-t)=\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2+\gamma_4\xi_1+\gamma_5\xi_2-r(T-t) \qquad (48)$$

Substituting (48) into (35) yields

$$E_p(S_1,S_2,t)=(2\pi i)^{-2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\left(\begin{array}{l}\dfrac{B(\xi_1,\xi_2)K^{(1+\xi_1+\xi_2)}}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)}\exp(\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2+\gamma_4\xi_1+\gamma_5\xi_2-r(T-t))\\ \times S_1^{-\xi_1}S_2^{-\xi_2}d\xi_1 d\xi_2\end{array}\right) \qquad (49)$$

Simplifying (49) further gives

$$E_p(S_1,S_2,t)=(2\pi i)^{-2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\left(\frac{B(\xi_1,\xi_2)Ke^{-r(T-t)}}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)}e^{(\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2+\ln K^{(\xi_1+\xi_2)}+\gamma_4\xi_1+\gamma_5\xi_2+\ln S_1^{-\xi_1}+\ln S_2^{-\xi_2})}d\xi_1 d\xi_2\right)$$

$$=\frac{Ke^{-r(T-t)}}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\left(\frac{B(\xi_1,\xi_2)}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)}e^{(\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2)}e^{(\xi_1\ln K+\xi_2\ln K+\gamma_4\xi_1+\gamma_5\xi_2-\xi_1\ln S_1-\xi_2\ln S_2)}d\xi_1 d\xi_2\right)$$

$$=\frac{Ke^{-r(T-t)}}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\left(\frac{B(\xi_1,\xi_2)}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)}e^{(\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2)}e^{[(\ln K-\ln S_1+\gamma_4)\xi_1+(\ln K-\ln S_2+\gamma_5)\xi_2]}d\xi_1 d\xi_2\right) \qquad (50)$$

where

$$\gamma_1\xi_1^2+\gamma_2\xi_1\xi_2+\gamma_3\xi_2^2=\gamma_1\xi_1^2+\frac{\gamma_2\xi_1\xi_2}{2}+\frac{\gamma_2\xi_1\xi_2}{2}+\gamma_3\xi_2^2=(\xi_1\quad\xi_2)\begin{pmatrix}\gamma_1 & \dfrac{\gamma_2}{2}\\ \dfrac{\gamma_2}{2} & \gamma_3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\Psi^T A\Psi \qquad (51)$$

with

$$\Psi^T=(\xi_1\quad\xi_2),\ A=\begin{pmatrix}\gamma_1 & \dfrac{\gamma_2}{2}\\ \dfrac{\gamma_2}{2} & \gamma_3\end{pmatrix},\Psi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

Also,

$$(\ln K-\ln S_1+\gamma_4)\xi_1+(\ln K-\ln S_2+\gamma_5)\xi_2=(\ln K-\ln S_1+\gamma_4\quad \ln K-\ln S_2+\gamma_5)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=B^T\Psi \qquad (52)$$

with

$$B=\begin{pmatrix}\ln K-\ln S_1+\gamma_4\\ \ln K-\ln S_2+\gamma_5\end{pmatrix}$$

Let

$$G(\Psi)=\frac{B(\xi_1,\xi_2)}{(\xi_1+\xi_2)(\xi_1+\xi_2+1)} \tag{53}$$

and

$$C=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{54}$$

Substituting (51), (52) and (53) into (50), yields

$$E_p(S_1,S_2,t)=\frac{Ke^{-r(T-t)}}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} G(\Psi)e^{\Psi^T A\Psi}e^{B^T\Psi}d\Psi \tag{55}$$

Since $\Psi \in C$, then let

$$\Psi = C+iX \Rightarrow d\Psi = idX \tag{56}$$

Substituting (56) into (55) leads to

$$\left.\begin{aligned} E_p(S_1,S_2,t) &= \frac{Ke^{-r(T-t)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(X)e^{(C^T AC+B^T C-X^T AX+i(B+2AC)^T X)}dX \\ &= \frac{Ke^{(-r(T-t)+C^T AC+B^T C)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(X)e^{-X^T AX}e^{i(B+2AC)^T X}dX \end{aligned}\right\} \tag{57}$$

where

$$X=\left(x_1,x_2\right)^T, G(X)=\frac{B(c_1+ix_1,c_2+ix_2)}{(c_1+ix_1+c_2+ix_2)(c_1+ix_1+c_2+ix_2+1)}$$

Diagonalizing the quadratic form $X^T AX$ using the transformation

$$X = \mathrm{T}X' \tag{58}$$

in order to simplify (57), where T is an orthogonal matrix given by

$$\left.\begin{array}{l} \mathrm{T}=\begin{pmatrix} \dfrac{-\gamma_2}{\left(2\zeta^2-2(\gamma_1-\gamma_3)\zeta\right)^{0.5}} & \dfrac{\gamma_1-\gamma_3-\zeta}{\left(2\zeta^2-2(\gamma_1-\gamma_3)\zeta\right)^{0.5}} \\ \dfrac{-\gamma_2}{\left(2\zeta^2+2(\gamma_1-\gamma_3)\zeta\right)^{0.5}} & \dfrac{\gamma_1-\gamma_3+\zeta}{\left(2\zeta^2+2(\gamma_1-\gamma_3)\zeta\right)^{0.5}} \end{pmatrix}, \\ \zeta=\left(\gamma_1^2+\gamma_2^2+\gamma_3^2-2\gamma_1\gamma_3\right)^{0.5} \end{array}\right\} \tag{59}$$

The quadratic form X^TAX *is* reduced to $X'^T \mathrm{A}X'$, where A is a diagonal matrix of eigenvalues given by

$$\lambda_1=\frac{\gamma_1+\gamma_2+\zeta}{2} \tag{60}$$

$$\lambda_2=\frac{\gamma_1+\gamma_2-\zeta}{2} \tag{61}$$

Therefore (57) becomes

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathrm{T}X')e^{-X'^T\mathrm{A}X'}e^{i(B+2AC)^T\mathrm{T}X'}dX' \tag{62}$$

Let $D=(B+2AC)^T\mathrm{T}$, then (62) yields

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathrm{T}X')e^{-X'^T\mathrm{A}X'}e^{iDX'}dX' \tag{63}$$

Setting

$$G(\mathrm{T}X')=R_1(X')+iR_2(X') \tag{64}$$

and using the fact that

$$e^{iDX'}=\cos DX'+i\sin DX' \tag{65}$$

Therefore (63) becomes

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(R_1(X')+iR_2(X'))(\cos DX'+i\sin DX')e^{-X'^T\mathrm{A}X'}dX'$$

Consider the real part of the expression $(R_1(X')+iR_2(X'))(\cos DX'+i\sin DX')$ in the above equation to get

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\text{Re}((R_1(X')+iR_2(X'))(\cos DX'+i\sin DX'))e^{-X'^T\text{A}X'}dX'$$

$$=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(R_1(X')\cos DX'-R_2(X')\sin DX')e^{-X'^T\text{A}X'}dX' \quad (66)$$

Using the transformation $X'=zu$, where $\text{u}=\left(\text{A}^{-1}\right)^{0.5}$, (66) becomes

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2\left(\det(\text{A})\right)^{0.5}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(R_1(z)\cos Dzu-R_2(z)\sin Dzu)e^{-z^Tz}dz \quad (67)$$

Setting $(R_1(z)\cos Dzu-R_2(z)\sin Dzu)=\Lambda(z)$ in (67), therefore (67) yields

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2\left(\det(\text{A})\right)^{0.5}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Lambda(z)e^{-z^Tz}dz \quad (68)$$

Equation (68) can be evaluated using a M-point Gauss-Hermite quadrature method in two dimensions. Hence the solution to the integral equation for the price of the European put option on a basket of two-dividend paying stocks is obtained as

$$E_p(S_1,S_2,t)=\frac{Ke^{(-r(T-t)+C^TAC+B^TC)}}{4\pi^2\left(\det(\text{A})\right)^{0.5}}\sum_{j=1}^{M}\sum_{k=1}^{M}\xi_j\xi_k\Lambda(z_j,z_k)$$

where ξ_i's are the weights of the Gauss-Hermite quadrature method,

$$B^T=\left(\ln K-\ln S_1-\left((r-q_1)-\frac{1}{2}\sigma_1^2\right)(T-t)\quad \ln K-\ln S_2-\left((r-q_2)-\frac{1}{2}\sigma_2^2\right)(T-t)\right)$$

$$A=\begin{pmatrix}\frac{1}{2}\sigma_1^2(T-t) & \frac{\rho\sigma_1\sigma_2(T-t)}{2}\\ \frac{\rho\sigma_1\sigma_2(T-t)}{2} & \frac{1}{2}\sigma_2^2(T-t)\end{pmatrix},\ C=\begin{pmatrix}c_1\\ c_2\end{pmatrix}\text{ and }C^T=\begin{pmatrix}c_1 & c_2\end{pmatrix}$$

This completes the proof.

4. 2. European basket put options Greeks

In financial mathematics, option sensitivities also known as Greeks describe the relationship between the value of an option and changes in one of its underlying parameters. They are easily obtained for plain vanilla put option with dividend paying stocks. From (20), write

$$E_p(S_1,...,S_n,t)=(2\pi i)^{-n}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}...\int_{c_n-i\infty}^{c_n+i\infty}\left\{\left(\frac{B(\xi_1,...,\xi_n)K^{\left(1+\sum_{i=1}^{n}\xi_i\right)}}{(\xi_1+...+\xi_n)(\xi_1+...+\xi_n+1)}\right)\times \left(e^{G(\xi_1,...,\xi_n)(T-t)}S_1^{-\xi_1}...S_n^{-\xi_n}\right)d\xi_1...d\xi_n\right\}$$

The above integral equation can be written in a compact form as

$$E_p(S_1,...,S_n,t)=e^{-r(T-t)}M^{-1}\left(\widehat{\phi}(\xi_1,...,\xi_n)Z(\xi^* j,T-t)\right) \tag{69}$$

with

$$G(\xi^*)=G(\xi_1,...,\xi_n)=-(\Phi(\xi^* j)+r) \tag{70}$$

and

$$\Phi(\xi^* j)=-\frac{1}{2}\sum_{j,k=1}^{n}\rho_{jk}\sigma_j\sigma_k\xi_j\xi_k+\sum_{j=1}^{n}\left((r-q_j)-\frac{\sigma_j^2}{2}\right)\xi_j \tag{71}$$

where $\widehat{\phi}(.)$ is the Mellin transform of the payoff function given by (29) and $Z(.)$ is the characteristic function of a multivariate Brownian motion with drift. By inducing appropriate derivative operator on the complex integral in (69), the following Greeks are obtained

i. Delta: This represents the rate of change between the option's price and the underlying asset price.

$$\Delta_E=-e^{-r(T-t)}M^{-1}\left(\frac{\xi_j}{S_j}\widehat{\phi}(\xi_1,\xi_2...,\xi_n)Z(\xi j,T-t)\right) \tag{72}$$

ii. Theta: This represents the rate of change between an option portfolio and time, or time sensitivity. In other words, it measures the sensitivity of the value of the derivative to the passage of time: the "time decay."

$$\Theta_E=-e^{-r(T-t)}M^{-1}\left((\Phi(\xi^* j)+r)\widehat{\phi}(\xi_1,...,\xi_n)Z(\xi j,T-t)\right) \tag{73}$$

iii. Rho: This measures sensitivity to the interest rate: it is the derivative of the option value with respect to the risk free interest rate (for the relevant outstanding term).

$$\rho_E=-(T-t)e^{-r(T-t)}M^{-1}\left(\sum_{j=1}^{n}(\xi_j-1)(T-t)\widehat{\phi}(\xi_1,\xi_2...,\xi_n)Z(\xi j,T-t)\right) \tag{74}$$

iv. Vega: This measures sensitivity to volatility. Vega is the derivative of the option value with respect to the volatility of the underlying asset.

$$\nu_E = \frac{\partial}{\partial \sigma_j}\left(e^{-r(T-t)} M^{-1}\left(\widehat{\phi}(\xi_1,\ldots,\xi_n) Z(\xi^* j, T-t)\right)\right) \tag{75}$$

Remark 4.3

i. Option sensitivities play a vital role for portfolio optimization and risk management.
ii. They have the ability to describe how vulnerable an option is to a particular risk factor.
iii. Gamma measures the rate of change in the delta with respect to changes in the underlying price. Gamma is the second derivative of the value function with respect to the underlying price.

5. NUMERICAL EXPERIMENTS AND CONCLUSION

In this section, some numerical experiments and discussion of results are presented as follows:

5. 1. Numerical experiments

EXPERIMENT 1

Consider the valuation of the European basket put option on two-dividend paying stocks with the following parameters:

$$\left.\begin{array}{l} S_1 = 50, S_2 = 50, K = \{60,70,80,90,100,110,120,130,140\}, \rho = 0.5, \sigma_1 = \sigma_2 = 0.2, \\ q_1 = q_2 = 0.05, r = 0.05, T = 1, M = 128, n_1 = n_2 = 1, c_1 = c_2 = 2 \end{array}\right\}$$

The results generated for the price of the European basket put option using the Mellin transform, Monte Carlo method with (10000 Monte Carlo trials) [13] and implied binomial tree method with (N=10 time steps) [13] are displayed in the Figures 1, 2, and 3 respectively. The comparative results analyzes of the three methods for the valuation of the European basket put option are displayed in the Figure 4 below.

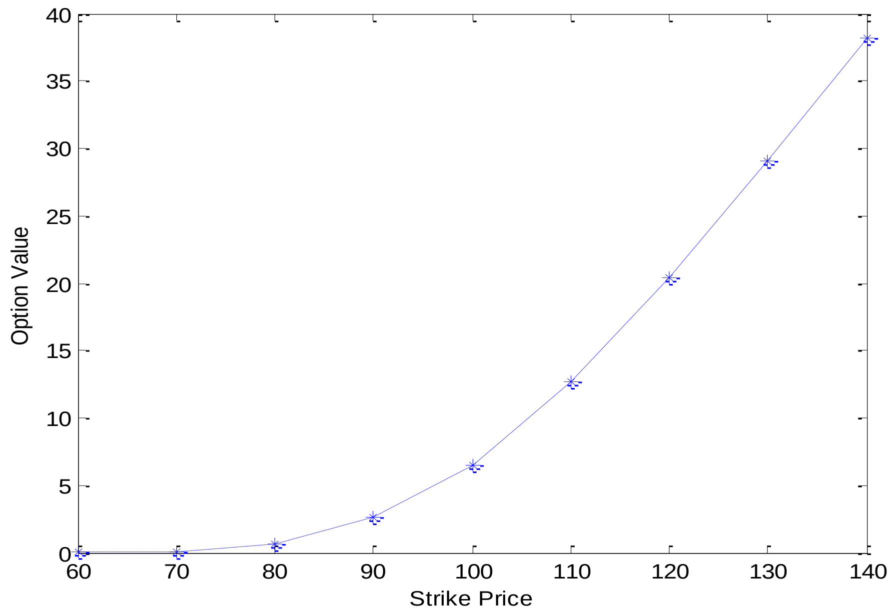

Figure 1. Price of European basket put using the Mellin transform in two dimensions

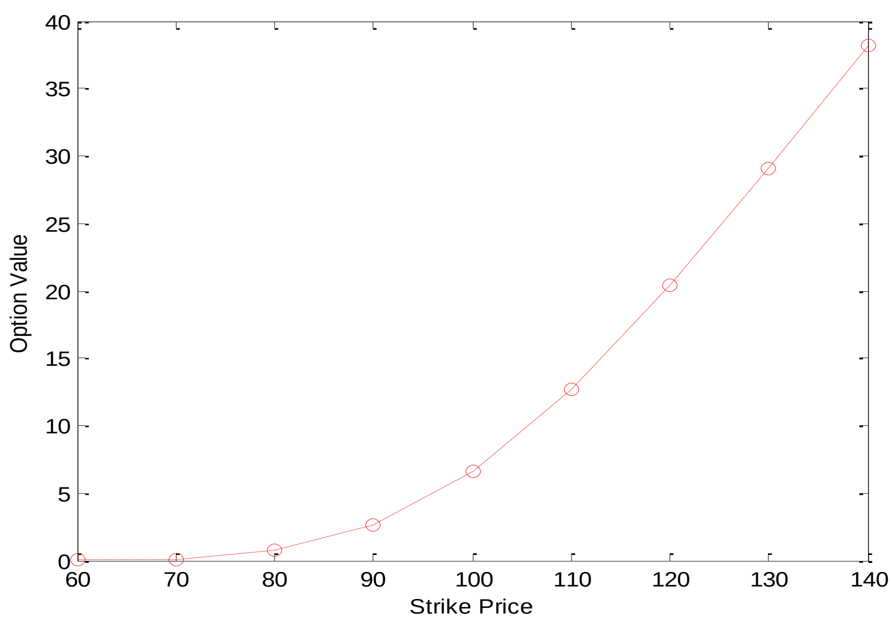

Figure 2. Price of European basket put using Monte Carlo method

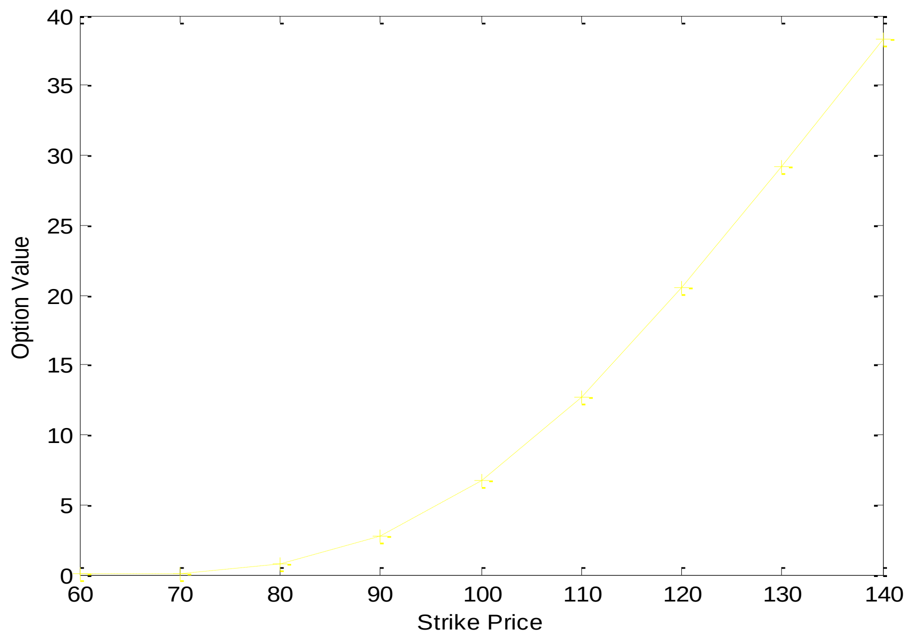

Figure 3. Price of European basket put using implied binomial tree model

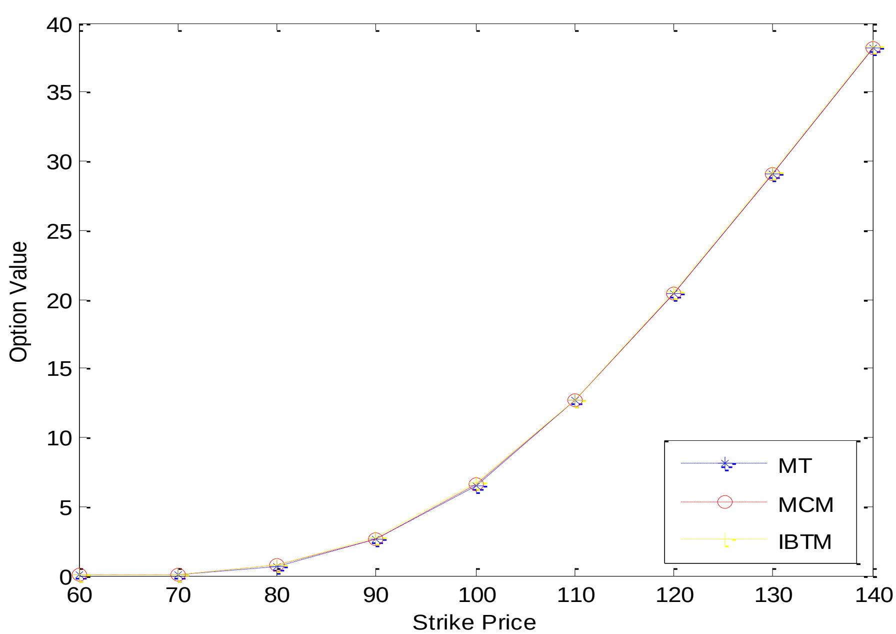

Figure 4. The comparative results analyzes of the Mellin transform in two dimensions, Monte Carlo method and implied binomial tree model for the valuation of the European basket put option

ANALYSIS OF EXPERIMENT 1

Figures 1, 2 and 3 show the performances of the Mellin transform in two dimensions, Monte Carlo method and implied binomial tree model for the valuation of the European put option on a basket of two-dividend paying stocks. Figure 4 compares the price of the European basket put option with two-dividend paying stocks via the Mellin transform with the price calculated using the Monte Carlo method and implied binomial tree model. It is observed from the Figure 4 that the Mellin transform performs very well and agrees with the values of the other two methods.

EXPERIMENT 2

Assume that the stocks are currently trading at $10 and $10 with annual volatilities of $\sigma_1 = 40\%$ and $\sigma_2 = 10\%, 20\%, 30\%$, respectively. The basket contains one unit of the first stock and one unit of the second stock. On January 1, 2014, an investor wants to buy a 1-year put option with a strike price of $20. The current annualized, continuously compounded interest rate is 3%. Use this data to compute the price of the European put basket option using the Mellin transform in two dimensions with $(c_1 = c_2 = 3, M = 128)$ and binomial (tree) model [12] varying the correlation coefficients $\rho = \{-0.5, 0.5\}$.

The comparative results analyzes of the two methods for negative and positive correlation coefficients are shown in the Tables 1 and 2 below respectively.

Table 1. The comparative results analyzes of the Mellin Transform in higher dimensions and binomial (tree) model with negative correlation coefficient.

σ_1	σ_2	Negative Correlation Coefficients, ρ	Binomial (Tree) Model	Mellin Transform in Two Dimensions
0.1	0.1	-0.5	1.108	1.104
0.1	0.2	-0.5	1.083	1.083
0.1	0.3	-0.5	1.198	1.198

Table 2. The comparative results analyzes of the Mellin Transform in higher dimensions and binomial (tree) model with positive correlation coefficient.

σ_1	σ_2	Positive Correlation Coefficients, ρ	Binomial (Tree) Model	Mellin Transform in Two Dimensions
0.1	0.1	0.5	1.496	1.494
0.1	0.2	0.5	1.783	1.782
0.1	0.3	0.5	2.101	2.100

The effect of the correlation coefficients on the price of the European basket put option with non-dividend paying stocks via the Mellin transform in two dimensions is displayed in the Figure 5 below.

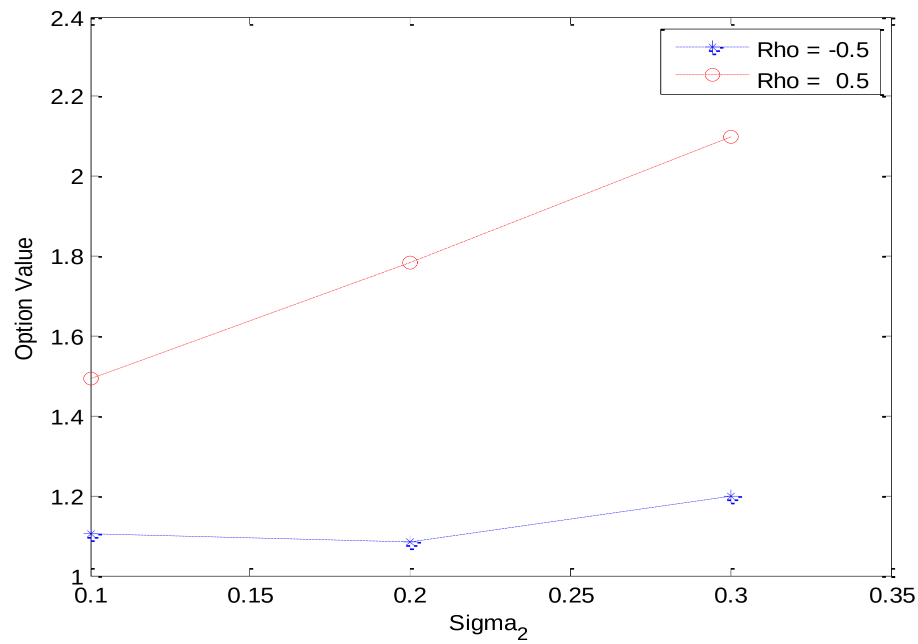

Figure 5. Effect of correlation coefficients on the price of European basket put option with $S_1 = S_2 = 10, K = 20, \rho \in \{-0.5, 0.5\}, \sigma_1 = 0.4, \sigma_2 \in \{0.1, 0.2, 0.3\}, q_1 = q_2 = 0, r = 0.03, T = 1,$ $M = 128, c_1 = c_2 = 3$ using the Mellin transform in two dimensions.

ANALYSIS OF EXPERIMENT 2

Tables 1 and 2 give the prices of the European basket put option as a function of the correlation coefficient ρ with non-dividend paying stocks using the Mellin transform in two dimensions and binomial (tree) model [12]. It is observed that the negatively correlated assets are more sensitive to correlation changes than positively correlated assets. This implies that sudden changes in the financial markets can lead to large losses for a basket option with well-diversified assets. It is observed from Figure 5 that the higher the correlation coefficient, the higher the price of the option. It is also observed from Figure 5 that the price of the European basket put option generated by the Mellin transform in two dimensions when the correlation coefficient $\rho = 0.5$ is greater than the price obtained when $\rho = -0.5$.

EXPERIMENT 3

Consider the valuation of European basket put option which pays three-dividend yields using the Triple Mellin Transform (TMT), Monte Carlo Method (MCM) with (10000 Monte Carlo trials) [13] and Implied Binomial Model with (N = 30 time steps) (IBM) [13] in the context of Black-Scholes-Merton Model (BSM) with the following parameters

Parameters	**Value**
Time to expiry	T = 12 months
Risk-neutral interest rate	r = 0.05
Dividend paying stocks,	$q_1 = 0.05, q_2 = 0.05, q_3 = 0.05$
Correlation	$\rho = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$
Underlying asset prices	$S_1 = S_2 = S_3 = 33.33$
Exercise prices	K = 60, 70, 80, 90, 100, 110, 120, 130, 140
Volatilities	$\sigma_1 = 0.2, \sigma_2 = 0.2, \sigma_3 = 0.2$

The comparative results analyses of the three methods against the Black-Scholes-Merton model are shown in the Figure 6 below. The absolute differences to the results from the Black-Scholes-Merton model are shown in Figure 7 below.

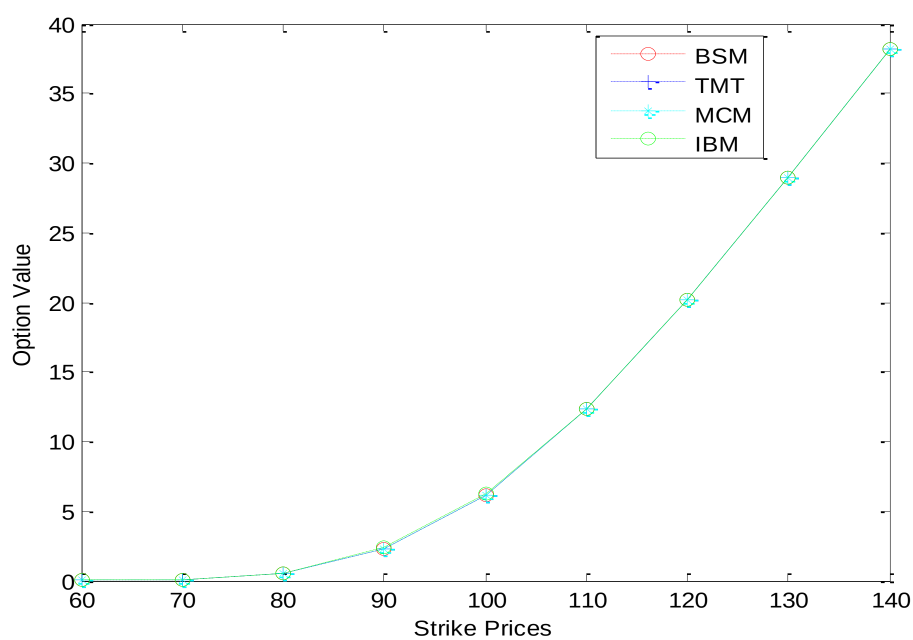

Figure 6. The comparative results analyses of the three methods against the Black-Scholes-Merton

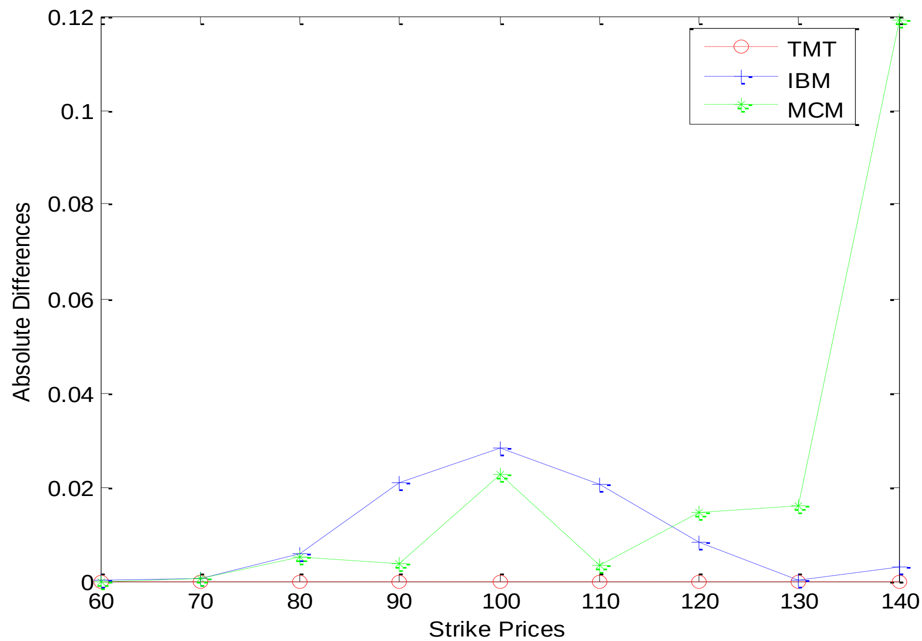

Figure 7. The absolute differences to the results from the Black-Scholes-Merton model

ANALYSIS OF EXPERIMENT 3

It is observed from Figure 6 that the three numerical methods perform very well. It is also observed from Figure 6 that the values of the triple Mellin transform coincide with the Black-Scholes-Merton values. Figure 7, it is observed that the triple Mellin transform has lower absolute differences to the results from Black-Scholes-Merton model.

5. 2. Conclusion

Most of the time, it is impossible to find closed form solutions for the values of some financial instruments. Finding alternative solutions for the governing pricing equations becomes therefore an appealing approach to pricing, especially since powerful desktop computers are now available. In this paper, the generalized Black-Scholes-Merton-like partial differential equation with multi-dividend paying stocks under geometric Brownian motion was derived. The multidimensional Mellin transform was applied to derive the integral equation for the price of the European put option on a basket of multi-dividend paying stocks. The numerical solution shows that the integral equation for the price of the European basket put option which pays dividend yields with $n = 2$ can be evaluated using the Gauss-Hermite quadrature method in two dimensions with M-point. One of the main advantages of this approach is that the analytic formula for the multi-asset option Greeks can be derived, which, unlike most other approximations, capture the effects that individual asset volatilities and correlations have on the hedge ratios.

The Mellin transform is a good alternative approach for the valuation of the European basket put option. Finally, some modifications and extensions of the approach used in this paper to study payoff modified options can be explored.

References

[1] Aniela, K.I. Numerical methods for pricing basket options, Department of Mathematics, The Ohio State University, 2004.

[2] Barraquand, J. and Martineau, D. Numerical valuation of high dimensional multivariate American securities. *The Journal of Financial and Quantitative Analysis*, 30 (1995) 383-405.

[3] Black, F. and Scholes, M. The pricing of options and corporate liabilities. *Journal of Political Economy* 81 (1973) 637-654.

[4] Fadugba, S.E. and Nwozo, C.R. Integral representations for the price of vanilla put options on a basket of two-dividend paying stocks. *Applied Mathematics*, 2015; 6: 783-792. http://dx.doi.org/10.4236/am.2015.65074

[5] Flajolet, P., Gourdon, X. and Dumas P. Mellin transforms and asymptotics: Harmonic sums. *Theoretical Computer Science* 1995, 144: 3-58.

[6] JU, N. Pricing Asian and basket options via Taylor expansion. *Journal of Computational Finance*, 2002, 5: 79-103.

[7] Manuge D.J. Basket option pricing and the Mellin transform. M.Sc. Thesis, The University of Guelph, Ontario, Canada, 2013.

[8] Nwozo, C.R. and Fadugba, S.E. Mellin transform method for the valuation of some vanilla power options with non-dividend yield. *International Journal of Pure and Applied Mathematics*, 2014; 96: 79-104. (dx.doi.org/10.12732/ijpam.v96i1.7)

[9] Vasilieva, O. (2009). A new method of pricing multi-options using Mellin transforms and integral equations. Master's Thesis in Financial Mathematics, School of Information Science, Computer and Electrical Engineering, Halmstad University, Sweden.

[10] Panini, R. and Srivastav, R.P. Option pricing with Mellin transforms. *Mathematical and Computer Modelling*. 2004; 40: 43-56. (DOI:10.1016/j.mcm.2004.07.008)

[11] Reed I.S. The Mellin type of double integral. *Duke Mathematical Journal.* 1944; 11: 565-572.

[12] Schneggenburger C, A Monte Carlo pricing tool for American basket put options on two assets, Diploma thesis in Mathematical Finance, Oxford University, 2002.

[13] Henry Wan, Pricing American-style basket options by implied binomial tree. Applied Finance Project, University of California, Berkeley, 2002.

[14] Xu, G. and Zheng, H. Basket options valuation for a local volatility jump-diffusion model with asymptotic expansion method. *Insurance: Mathematics and Economics*, 2010; 47: 415-422.