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$D = 4, \ N = 1$ supergravity in superspace: general overview and technical analysis

by

Paolo Di Sia
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Paolo Di Sia  
University of Padova, Stradella S. Nicola 3, 36100 Vicenza, Italy  
E-mail address: paolo.disia@gmail.com

ABSTRACT  
Considering the supersymmetry, the Einstein theory of general relativity brings to supergravity and the superspace gives a geometrical meaning to the supersymmetry transformations. I consider in this work the technical complete construction of \( D = 4, \ N = 1 \) supergravity in a geometrical way, i.e. using superforms in superspace as extension of spinor-tensor calculus. Starting by the pure \( D = 4, \ N = 1 \) supergravity, the coupling with scalar multiplets (multiplets of Wess-Zumino) and vector multiplets is performed. I use the concepts of supersymmetry, superspace and rheonomic principle. Bianchi identities are analyzed and resolved, ending with the Bianchi identity of gravitino. Supergravity theories are the effective theories of superstring theories.

Keywords: Supersymmetry, Superspace, Rheonomy, Supergravity, Differential geometry, Calculus with forms, Scalar (Wess-Zumino) multiplets, Vector multiplets, Superstrings

Reviewer  
Prof. Ignazio Licata  
Institute for Scientific Methodology, Palermo, Italy
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PREFACE

This work is a both introductory and advanced exposure of the supergravity theory in four dimensions with a supersymmetric charge. Many are the reasons for the depth study of supergravity; I remember in particular:

a) on a hand, in the attempt to explain the structure of fundamental interactions of Nature, the reconciliation of quantum mechanics with general relativity is a significant logical necessity. An accredited candidate for a quantum theory of gravitation is the superstring theory, whose low-energy approximation is supergravity;

b) on the other hand, Higgs fields are crucial to explain the spontaneous symmetry breaking; a very satisfactory solution to the problem of the gauge hierarchy appears to be the spontaneously broken local supersymmetry.

Considering the two opposite sides of the energy scale, we arrive at the same suggestion: the phenomenology of particle physics can be described in terms of an effective supergravity model.

It is also a fact that supersymmetry is a deep symmetry principle, mathematically very elegant, with big implications. It has the same value of the principle of general covariance and similarly provides an accurate basis for the formulation of the laws of Nature. The presented mathematics uses a language that emphasizes the underlying geometric structure.

The work consists of six chapters.

After a general introduction on the supersymmetric theory of particles, on the meaning and importance of supergravity in itself and as “effective theory” of superstring (Chapter I), I proceed to the technical study of pure supergravity in four dimensions with a supersymmetric charge (Chapter II). Supergravity is extended to superspace and it is built, using the rheonomic principle and Bianchi identities. Chapter III is devoted to the differential geometry of Kähler manifolds. In Chapter IV the pure $D = 4, N = 1$ supergravity is coupled to $n$ scalar multiplets. In Chapter V the vector multiplets are also coupled. Chapter VI ends with the conclusions.

After this I present a rich appendix about the algebra of gamma matrices, useful formulas for the development of algebra of gamma matrices, irreducible basis of $D = 4, N = 1$ superspace, Fierz formulas and self-duality identities, two tables showing the characteristics of de Sitter and Poincarè $D = 4, N = 1$ supergravity and $D = 4, N = 1$ supergravity coupled to $n$ scalar multiplets.
Chapter I

INTRODUCTION

1.1. Supersymmetric theory of particles

Supersymmetry (briefly called also SUSY) is a 1974-discovered symmetry, which has attracted high attention in physics and mathematics. About the main reasons:

a) it is a new symmetry, and history teaches that considerations about symmetry led to progress in fundamental and theoretical physics;

b) it is characterized by changing bosons into fermions and vice-versa, therefore it represents very different properties with respect to those arising from the ordinary symmetries of high energy physics.

Its simplest version is the $N = 1$ supersymmetry, with $N$ number of supersymmetry generators, and supergravity (briefly called also SUGRA) is the local version of the supersymmetric theory. To understand how supergravity theory fits with the phenomenology of elementary particles, we know that such phenomenology is well described by the “standard model” [1].

The standard model is based on SU(3) × SU(2) × U(1) gauge group for strong, weak and electromagnetic interactions. It contains 12 gauge bosons with spin 1: 8 gluons of SU(3), 3 weak gauge bosons of SU(2) and the gauge hypercharge boson of U(1). The photon is a particular combination of a gauge boson of SU(2) and the hypercharge boson. Fermions of the theory are three generations of quarks and leptons (Figure 1).

Figure 1. Elementary particles of standard model [2].
Figure 2 presents a scheme of interactions among particles described by the standard model. With this model it is correctly described the physics of particles up to the energy region around 100 GeV.

At theoretical level, the standard model is not based on an effective Lagrangian, such as the Fermi theory of weak interactions, but it is a renormalizable field theory. It refers to unified theories, in which the gauge group SU(3) x SU(2) x U(1) is unified in larger groups, such as SU(5) to this mass scale, or SO(10), or larger groups, such as E_6 [3,4].

Assuming that the standard model is valid up to a scale of unification of $10^{15}$ GeV or more, the scale of weak interactions of 100 GeV is very small compared with the previous one and with the Planck scale $10^{19}$ GeV. Considering these three scales as “input” parameters of the theory, the square mass of the scalar particles in the Higgs sector should be chosen with accuracy of order of $10^{-34}$, compared to the Planck mass.

Theories with an adjustment of such accuracy are also called “non-natural”. The way to make “natural” such a theory can be a symmetry that implies that the small parameters are exactly zero and their actual values are related to the breaking of this symmetry. Supersymmetry possesses this characteristic, it makes “natural” the standard model.

![Figure 2. Scheme of interactions among particles described by the standard model [5].](image)

If we want to introduce supersymmetry in the standard model, next to each boson (fermion) of the model, it must be introduced a fermionic (bosonic) supersymmetric partner (called also “superpartner” or “sparticle”) (Figure 3) and for having acceptable phenomenological models there is the need of an additional Higgs supermultiplet.

If supersymmetry is a local symmetry, then it necessarily includes gravity; we have therefore the “supergravity” (Figure 4) [6,7].
1.2. Models of supergravity

Supergravity models have a higher predictive power than those based on global supersymmetry, because they allow to solve also the problem of the “gauge hierarchy” of standard model.

Supersymmetry can be applied to particle physics if it is broken. If not, fermions and bosons would have the same mass and this goes against experimental data. If supersymmetry is spontaneously broken, we have a “mass splitting” between fermions and bosons belonging to the same multiplet. One of the major obstacles encountered in the construction of unification theories with global supersymmetry was the fact that such “mass splitting” occurred in a wrong way. Even as a result of supersymmetry breaking, with consequent diversification of the mass between fermions and bosons, the mass relation of the supertrace of $M^2$:

$$\text{Str } M^2 = \Delta m^2,$$  \hspace{1cm} (1)

**Figure 3.** A list of sparticles.
valid in absence of matter, still holds and this is in contradiction with experimental results. In local supersymmetry theories, as supergravity, Eq. (1) becomes:

$$\text{Str } M^2 = \Delta m^2, \quad (2)$$

with $\Delta m^2$ linear combination of the square mass of gravitino and of $D^a D^a$, with $D^a$ auxiliary field of the vector multiplet. This discovery opened the way to the application of spontaneously broken $\mathcal{N} = 1$ supergravity to the description of the phenomenology of particles.

### 1.3. $\mathcal{N} = 1$ supergravity

It is possible to create supergravity models in four dimensions with more than one supersymmetry charge, but by a phenomenological viewpoint the $\mathcal{N} = 1$ theory has an interesting peculiarity: the spectrum of known fermions in the region of 100 GeV implies that they are subject to complex representations of the gauge group. This is not compatible with the extended $N > 1$ supersymmetry, which allows real representations of the particles with respect to the considered gauge groups. With $N = 1$ supersymmetry we can work on the contrary with complex chiral representations of particles.

Supersymmetric theories are invariant with respect to a set of transformations that change the spins of particles of half a unit, transforming bosons into fermions and vice-versa. These supersymmetry transformations are generated by a Majorana spinor $Q$, that satisfies the following algebra:

$$[Q, P_\mu] = 0, \quad \{Q, \bar{Q}\} = -2 \gamma_\mu P^\mu, \quad (3)$$

with $P^\mu$ the translation generator.

If supersymmetry is local, even translations are locally realized. An invariance with respect to local translations is substantially equivalent to an invariance with respect to the general coordinates transformations. Therefore an invariant theory with respect to local transformations of supersymmetry necessarily includes gravity.

Supersymmetry has been considered for many years as a very interesting mathematical structure from the point of view of quantum field theory, but not for particle physics level. The supersymmetry algebra appears formally as a possible extension of Lorentz and Poincarè
algebras, which are the basis of the relativity theory. The supersymmetry generators, acting on ordinary fields, create new fields.

Supersymmetry does not connect directly together the known particles; this is only an apparent problem, because it has been postulated the existence of new particles, which are the superpartners of ordinary ones. The theory is formulated by coupling the gravitational multiplet, containing the graviton of spin 2 and its spin 3/2 spinorial partner (gravitino), to matter supermultiplets of mass zero. They are of two types:

i) the vector supermultiplets, that have an index in the adjoint representation of the gauge group;
ii) the chiral (Wess-Zumino) supermultiplets, constituted by a Majorana spinor of spin 1/2 and by a complex scalar.

After the spontaneous breaking of gauge invariance, some of gauge vector multiplets acquire mass, while the corresponding Wess-Zumino fields are deleted (Table 1).

There is no conflict, but complementarity between supersymmetry and grand unification approaches. The grand unification aims at a unified description of electromagnetic, weak and strong interactions, while supersymmetry seems the natural structure for the introduction of gravity.

1.4. Supergravity as “effective theory” of superstring

Table 1. Particle content of a spontaneously broken supersymmetric gauge theory.

<table>
<thead>
<tr>
<th>MULTIPLETS</th>
<th>SPIN 1</th>
<th>SPIN 1/2</th>
<th>SPIN 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>VECTOR MULTIPLETS</td>
<td>Massless gauge vector multiplets</td>
<td>Photon</td>
<td>Photino</td>
</tr>
<tr>
<td>(1, 1/2)</td>
<td>Gluons</td>
<td>Gluinos</td>
<td></td>
</tr>
<tr>
<td>SCALAR MULTIPLETS</td>
<td>Massive gauge vector multiplets</td>
<td>Intermediate gauge bosons</td>
<td>Heavy fermions</td>
</tr>
<tr>
<td>(1/2, 0+, 0−)</td>
<td>Matter Wess-Zumino multiplets</td>
<td>Leptons</td>
<td>Squarks</td>
</tr>
</tbody>
</table>

Apparently the standard model including supergravity seems irrelevant by a physical point of view, because it is not renormalizable. Field theories including gravity are in fact not renormalizable and this is also the case of \( N = 1 \) supergravity. Initially it was hoped that the existence of supersymmetry could lead to cancellations of infinities of quantum theory; this is true at the first perturbation order, but in general is not possible to get a fully finite theory. The problem of renormalizability of supergravity disappears if it is not considered as “fundamental theory”, but as “effective theory” of a superstring theory.
For “effective theory” of superstring we mean the theory obtainable integrating all massive modes of string theory in the path integral. In general, this theory contains higher order derivatives, whose scale is fixed by the $\alpha'$ string constant. If we consider the terms which do not contain more than two derivatives of fields, it is possible to proof that supergravity theories are the effective theories derived by superstring theories (Figure 5) [1,8-17].

Figure 5. Supergravity as intermediate step between superstrings and supersymmetric standard model.
Chapter II

$D = 4, N = 1$ PURE SUPERGRAVITY AND RHEONOMY

2.1. Introduction

Gravity is the gauge theory of the Poincare group $\text{ISO}(1,3)$ in a four-dimensional (4-D) spacetime; supergravity is its supersymmetric extension. The Einstein-Cartan action can be written in the form:

$$ A = \int_{M_4} R^{ab} (\omega) \wedge V^c \wedge V^d \varepsilon_{abcd} \quad (4) $$

where $M_4$ is a 4-D Riemannian manifold and $R^{ab}$ is the 2-form of curvature:

$$ R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb} \quad (5) $$

and $V^a$ is the vierbein. Action (4) is equivalent to the action of gravity written in the tensor formalism:

$$ R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = -4 R^a_{\mu\nu} \det V \, d^4 x, \quad (6) $$

where $R^a_{\mu\nu} = R^m_{\mu\nu} = R$ is the scalar of curvature, $\det V = \sqrt{-g} = \sqrt{-\det g_{\mu\nu}}$, Latin indices = flat indices, Greek indices = curved indices. Therefore it is:

$$ \int_{M_4} R^{ab} (\omega) \wedge V^c \wedge V^d \varepsilon_{abcd} = -4 \int_{M_4} R \sqrt{-g} \, d^4 x. \quad (7) $$

We have two gauge fields: the spin connection $\omega^{ab}$ and the vierbein $V^a$:

$$ \omega^{ab} = \omega^a_\mu \, dx^\mu, \quad (8) $$

$$ V^a = V^a_\mu \, dx^\mu. \quad (9) $$

In the first order formalism both gauge fields are treated as independent. The couple $\{V^a, \omega^{ab}\}$, considered as a single entity, is a multiplet in the adjoint representation of the Poincare group:

$$ \mu^a (x) T^A = \omega^{ab} (x) J_{ab} + V^a (x) P_a. \quad (10) $$

$\mu^a (x) = \mu^a_\mu (x) \, dx^\mu$ is the gauge field of the Poincarè group, $J_{ab}$ and $P_a$ are respectively the generators of the Lorentz transformations and of four-dimensional translations. The strength of the field associated with $\mu^a$ is defined as the “Poincarè Lie algebra-valued” 2-form of curvature:
\[ R^A = d\mu^A + \frac{1}{2} C^A_{\ BC} \mu^B \wedge \mu^C. \] (11)

Dividing the \( A \) index as \((ab, a)\), we have:

\[ R^{ab} = d\omega^{ab} - \omega^b_c \wedge \omega^c_b; \] (12)

\[ R^a = dV^a - \omega^a_b \wedge V^b \equiv \mathcal{D} V^a. \] (13)

The associated Bianchi identities are given by:

\[ \mathcal{D} R^{ab} = 0, \quad (14) \]

\[ \mathcal{D} R^a + R^{ab} \wedge V_b = 0. \quad (15) \]

Therefore the Lorentz algebra-valued curvature is the field strength of the spin connection, while the vector-valued curvature, or torsion, is the field strength of the vierbein field. The Einstein-Cartan action is invariant under general transformations of coordinates generated by the Lie derivatives:

\[ l_x \omega^{ab} = \omega^{ab}(x) - \omega^{ab}(x) = (\mathcal{D} + d \mathcal{E}) \omega^{ab}; \quad (16) \]

\[ l_x V^a = V^a(x) - V^a(x) = (\mathcal{E} + d \mathcal{E}) V^a. \quad (17) \]

Since the action (4) is written using only external products of forms and external derivatives \( "d" \), the invariance under diffeomorphisms is guaranteed by the general law of transformation of forms under diffeomorphisms. Using the definitions of curvatures, the Lie derivatives (16) and (17) can be written in the form:

\[ l_x \omega^{ab} = \mathcal{D} \mathcal{E}^{ab} + \mathcal{E} R^{ab}; \quad (18) \]

\[ l_x V^a = \mathcal{D} \mathcal{E}^a + \mathcal{E} \omega^{ab} V_b + \mathcal{E} R^a. \quad (19) \]

The gauge transformations of the Poincarè group are given by:

\[ \delta^\text{GAUGE}_\varepsilon \omega^{ab} = \mathcal{D} \varepsilon^{ab}, \quad (20) \]

\[ \delta^\text{GAUGE}_\varepsilon V^a = \mathcal{D} \varepsilon^a + \varepsilon^{ab} V_b, \quad (21) \]

and the Lorentz gauge transformations result from Poincarè transformations with \( \varepsilon^a = 0 \).

Varying action (4) with respect to the vierbein field, we get the Einstein equation of pure gravity:

\[ R^a_{\ \ b} - \frac{1}{2} \varepsilon^a_{\ b} R = 0, \quad (22) \]
and from the variation of $\omega^{ab}$:

$$R^c \wedge V^d \varepsilon_{abcd} = 0 \rightarrow R^c = 0. \quad (23)$$

Now we extend this formalism by coupling the Lagrangian describing pure gravity with the Lagrangian describing the Rarity-Schwinger field with spin 3/2; this is because constructing Lagrangians invariant under local supersymmetry transformations, the “gauging” of the supersymmetry transformations necessarily involves the gauge field of supersymmetry:

$$\psi = \psi^\alpha Q_\alpha = \psi^\alpha (x) dx^\alpha Q_\alpha, \quad (24)$$

where $\psi^\alpha (x)$ describes a massless particle with spin 3/2 in 4-D and $Q_\alpha$ is the supersymmetry generator. Such field, partner of graviton, is the “gravitino” and the corresponding interacting theory is $N = 1, D = 4$ supergravity.

We consider the Lagrangian of minimal coupling for the Rarita-Schwinger field. The Lagrangian of free field in Minkowski space is:

$$\mathcal{L}_{\text{R.S.}} = \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma, \quad (25)$$

with $\psi_\mu$ satisfying the Majorana condition $\psi^+ \gamma^0 = \psi C$. $\psi_\mu$ contains a part with spin 3/2 and a part with spin 1/2. The second one can be eliminated by fixing the gauge choice $\gamma^\mu \psi_\mu = 0$. The motion equation following by $\mathcal{L}_{\text{R.S.}}$ is:

$$\varepsilon^{\mu\nu\rho} \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma = 0, \quad (26)$$

The minimal Lagrangian coupled to gravity is obtainable through the substitutions:

$$\partial_\mu \rightarrow \mathcal{D}_\mu (\omega), \quad (27)$$

$$\gamma_\mu \rightarrow \gamma_a V^a_\mu, \quad (28)$$

$$\mathcal{D}_\mu (\omega) \lambda = \partial_\mu \lambda - \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \lambda, \quad (29)$$

for a generic spinorial field $\lambda$. The complete action describing the coupling is:

$$A = \int_{M_4} -4 R \sqrt{-g} d^4 x + 4 \bar{\psi}_\mu \gamma_5 \gamma_\alpha \mathcal{D}_\nu \psi_\rho \psi^\alpha V^\nu_\lambda \varepsilon^{\mu\nu\rho\lambda} d^4 x =$$

$$= \int_{M_4} R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4 \bar{\psi}_\mu \gamma_5 \gamma_\alpha \mathcal{D}_\nu \psi \wedge V^\nu. \quad (30)$$

Varying action (30) with respect to $\omega^{ab}_\mu$, we get:

$$2 R^c \wedge V^d \varepsilon_{abcd} = 0 \rightarrow R^c = 0, \quad (31)$$
with $R^c$ defined by:

$$R^c = \mathfrak{D} V^c - \frac{i}{2} \bar{\psi} \gamma^c \psi. \quad (32)$$

Varying $V^a$ and $\psi$ we get respectively:

$$2 R^{ab} \wedge V^c \varepsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_d \mathfrak{D} \psi = 0, \quad (33)$$

$$8 \gamma_5 \gamma_a \mathfrak{D} \psi \wedge V^a - 4 \gamma_5 \gamma_a \psi \wedge R^a = 0. \quad (34)$$

The Langragian (30) is invariant under local Lorentz transformations, spacetime diffeomorphisms and with respect to new transformations containing an anti-commutative parameter $\varepsilon$, i.e. the supersymmetry transformations. They are:

$$\delta_{\varepsilon} V^a = i \bar{\psi} \gamma^a \psi, \quad (35)$$

$$\delta_{\varepsilon} \psi = \mathfrak{D} \varepsilon, \quad (36)$$

$$\delta_{\varepsilon} \omega^{ab} \text{ (first order)} = -2 \varepsilon^{abrs} \bar{\psi} \gamma_5 \gamma_m \rho_m V^m - 2 \varepsilon^{tr[a} \bar{\psi} \gamma_5 \gamma_5 \rho_m V^b), \quad (37)$$

2.2. Supergravity in superspace

In order to give a geometrical meaning to supersymmetry transformations, the spacetime fields $V^a_\mu$, $\psi_\mu$, $\omega^{ab}_\mu$ are interpreted as 1-forms in superspace. In this way the transformations of supersymmetry can be interpreted as Lie derivatives in superspace.

The 1-forms $(V^a, \psi)$ can be considered as a single object $E^a = (V^a, \psi)$, called in general “supervielbein” (supervierbein in 4-D). $(V^a, \psi)$ form a basis in the cotangent plane in a point $P$ of the superspace. Supergravity can be naturally interpreted as a theory in superspace. The superspace structure equations define the curvatures:

$$R^{ab} = d\omega^{ab} - \omega^c \wedge \omega^{cb} \equiv \mathcal{R}^{ab}, \quad (38)$$

$$R^a = \mathfrak{D} V^a - \frac{i}{2} \bar{\psi} \gamma^a \psi, \quad (39)$$

$$\rho = \mathfrak{D} \psi, \quad (40)$$

where now $\omega^{ab}$, $V^a$, $\psi$ are 1-forms in superspace and $R^{ab}$, $R^a$, $\rho$ are the corresponding curvatures. In compact notation we can write:

$$R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C, \quad (41)$$
with:

\[ R^a = (R^{ab}, R^a, \rho). \]  \hspace{1cm} (42)

The Bianchi identities associated to curvatures (38-40) are:

\[ D^a R^{ab} = 0 , \]  \hspace{1cm} (43)

\[ D^a R^a + R^{ab} \wedge V_b - i \bar{\psi} \wedge \gamma^a \rho = 0 , \]  \hspace{1cm} (44)

\[ D^a \rho + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi = 0 . \]  \hspace{1cm} (45)

2.3. The rheonomic principle

It is assumed that the fields \( \omega^{\mu}_{\nu}, V^a, \psi \) introduced for the spacetime description of supergravity are the same fields that enter in the structure equations of superspace (38-40). The 1-forms \( \mu^a = (\omega^{ab}, V^a, \psi) \) are defined on \( M^{4|4} \), or \( \tilde{G} \), which has a fiber bundle structure:

\[ \tilde{G} = \tilde{G}(M^{4|4}, SO(1,3)) \]  \hspace{1cm} (46)

while the fields of supergravity in the standard approach of components are defined only on spacetime \( M^4 \). This requires that the spacetime fields \( \omega^{ab}_{\mu}(x), V^a_{\mu}(x), \psi^a_{\mu}(x) \) are interpreted as the spacetime boundary values of the superfields of superspace \( \mu^a = \mu^a(x, \theta) \):

\[ V^a = V^a_{\mu}(x) dx^\mu \rightarrow V^a(x) = V^a(x, \theta)_{\theta^a = 0} , \]  \hspace{1cm} (47)

\[ \psi = \psi^a_{\mu}(x) dx^\mu \rightarrow \psi(x) = \psi(x, \theta)_{\theta^a = 0} ; \]  \hspace{1cm} (48)

\[ \omega^{ab} = \omega^{ab}_{\mu}(x) dx^\mu \rightarrow \omega^{ab}(x) = \omega^{ab}(x, \theta)_{\theta^a = 0} . \]  \hspace{1cm} (49)

They are the values in \( \theta^a = 0 \) of the restriction on bosonic cotangent plane of the corresponding 1-forms in superspace. The mapping:

\[
\text{rh: } \begin{cases}
V^a(x) \rightarrow V^a(x, \theta) \\
\psi(x) \rightarrow \psi(x, \theta) \\
\omega^{ab}(x) \rightarrow \omega^{ab}(x, \theta)
\end{cases}
\]  \hspace{1cm} (50-52)

is called "rheonomic extension mapping". It is assumed that the "outer" components \( R^a_{\alpha L} \) can be algebraically expressed in terms of the purely spacetime (or "inner") components \( R^a_{\mu \nu} \).
\[ R^A_{\alpha L} = C^{A/\mu \nu}_{\alpha L/\beta} R^\beta_{\mu \nu}. \] (53)

\( C^{A/\mu \nu}_{\alpha L/\beta} \) are constant tensors, \( \mu \) and \( \nu \) indices of bosonic spacetime coordinates, \( \alpha \) a spinor index associated to coordinate \( \theta^a \), \( L=(\alpha, \mu) \), \( A \) and \( B \) are indices of the Lie superalgebra. In the presence of rheonomic constraints, the physical content of a superspace theory is completely determined by means of a purely timespace description (Table 2).

**Table 2.** Physical content of a superspace theory in the presence of rheonomic constraints.

Then a transformation of supersymmetry can be identified with the Lie derivative \( l_{\xi} \) in the presence of rheonomic constraints. In general the transformations of supersymmetry do not close “off-shell” an algebra of transformations. In order that the Lie derivatives close the algebra, it is necessary that the operator of external derivative used in the definition satisfies the condition \( d^2 = 0 \), or, equivalently, that the supercurvatures satisfy the Bianchi identities in superspace.

This generally does not happen, because the “out” components of curvatures must satisfy the rheonomic constraints. It is possible to show that the closure of Bianchi identities takes place only if the “in” curvatures satisfy algebraic and differential relations, which can be interpreted as the equations of motion on spacetime.

In conclusion, the algebra of supersymmetry is closed only if the equations of motion are satisfied. In this case it says that the theory is supersymmetric “on-shell”. It is possible to explicitly verify that the rheonomic constraints imply the equations of motion of graviton and gravitino on spacetime.

**2.4. Extension of the action principle**

Usually the action is written as follows:

\[ S = \int_{\Omega} \mathcal{L}(\phi), \] (54)
where $\Omega$ is an $n$-dimensional manifold and $\mathcal{L}$ the scalar density. The action $S = S[\phi]$ is thus a functional of the fields configurations only. A natural generalization of (54) is obtained by considering as Lagrangian $\mathcal{L}(\phi)$ forms of grade $D < n$:

$$S = S[\phi, M_D] = \int_{M_D} \mathcal{L}(\phi), \quad (55)$$

where $M_D$ is a $D$-dimensional sub-manifold of $\Omega$. The action thus becomes a function of both the field configurations $\phi(x)$ and the $M_D$ surface. Asking that $S$ is minimal with respect both to variations of fields and of the surface, we get the classical equations of motion. In general these equations are rather complicated and can be non-local. It is possible to overcome these difficulties if we suppose that the fields $\{\phi_i\}$ are a set of external forms of various grades $p$ and that the Lagrangian is obtained by $\{\phi_i\}$ using only the diffeomorphism-invariant operations of the outer algebra, i.e.:

- the outer derivative “$d$”: $\phi \rightarrow d\phi$,
- the wedge product “$\wedge$”: $(\phi_1, \phi_2) \rightarrow \phi_1 \wedge \phi_2$,
- the exclusion of the Hodge operator of forms dualization “*”.

In this way every deformation of the surface $M_D$ can be compensated by a diffeomorphism in fields superspace $\{\phi_i\}$. This implies that the full set of variational equations associated to the new action is given by the usual equations of motion obtained by varying that with respect to the fields “on a fixed surface”: $\delta S / \delta \phi, \mathcal{L} = 0$.

Being written with forms, these equations hold not only on the surface of integration, but on the whole superspace $\Omega$. We say that the action is “geometric”, i.e. is built with external forms, using the diffeomorphism -invariant operations “$d$” e “$\wedge$”. The exclusion of the Hodge's duality operator “*” implies the use of the first order formalism for all fields; the derivative $\phi_\theta$ of a scalar field $\phi$ or the field strength $F_{mn}$ of a spin 1 field $A$ are introduced as independent objects and are varied independently in the action.

2.5. Rheonomy, $D = 4$, $N = 1$ supergravity and “on-shell” supersymmetry

We can integrate the Lagrangian (30) on a 4-dimensional bosonic surface immersed in the superspace and adopt this as geometric Lagrangian of $N = 1$ supergravity:

$$\mathcal{L}_{\text{space-time}} (V^a (x, 0)|_d\theta = 0, \omega^{ab} (x, 0)|_d\theta = 0, \psi(x, 0)|_d\theta = 0) \rightarrow \mathcal{L}_{\text{superspace}} (\mu^a (x, \theta)). \quad (56)$$

The extended action is:

$$A_{\text{extended}}^{D=4,N=1} = \int_{M_4 \subset R^{4|4}} (R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4 \overline{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a), \quad (57)$$

with $R^{4|4}$ superspace manifold and $M_4$ 4-dimensional bosonic surface that, for example in $\theta = 0$, we can identify as that of spacetime. The equations of motion have the same form of those of spacetime approach, but now they hold on the entire superspace $R^{4|4}$:

$$2 R^c \wedge V^d \epsilon_{abcd} = 0, \quad (58)$$
For analyzing Eqs (58-60), we expand them in a complete basis of 3-forms:

\[ R^{ab} = R_{cd}^{ab} V^c \wedge V^d + \bar{\theta}_e^{ab} \psi \wedge V^c + \bar{\psi} \wedge K^{ab} \psi, \quad (61) \]

\[ R^a = R_{bc}^a V^b \wedge V^c + \bar{\theta}_c^a \psi \wedge V^c + \bar{\psi} \wedge K^a \psi, \quad (62) \]

\[ \rho = \rho_{ab} V^a \wedge V^b + H_c \psi \wedge V^c + \Omega_{e\beta} \psi^\alpha \wedge \psi^\beta, \quad (63) \]

with \( \theta_e^{ab} \) and \( \theta^a_c \) Majorana spinor-tensors, \( K^{ab} = -K^{ba} \), \( K^a \subset H^{c} \) Majorana-spinor-valued 4x4 matrices in spinorial space, \( \Omega_{e\beta} \) Majorana-spinor-valued matrix in the same space. We cancel now the coefficients of all independent 3-forms.

The \( V \wedge V \wedge V \) projection gives the propagation equations for the spacetime components \( R_{cd}^{ab}, R^a_{bc}, \rho_{ab} \):

\[ R^a_{pq} = 0, \quad (64) \]

\[ 8 \gamma_5 \gamma_d \mathcal{D}_b \psi \epsilon^{abcd} = 0, \quad (65) \]

\[ -8 (R^{a_0}_{q_0} - \frac{1}{2} \delta^{a_0}_{q_0} R^a_{q_0}) = 0, \quad (66) \]

that hold on whole \( R^{4|4} \). Their restrictions on \( M^4 \) are the ordinary spacetime equations. From \( \psi \wedge \psi \wedge \psi, \psi \wedge \psi \wedge V, \psi \wedge V \wedge V \) projections, we get:

\[ K^{ab} = K^a = \Omega_{e\beta} = \bar{\theta}_c^a = H_c = 0, \quad (67) \]

\[ \bar{\theta}_{d_0}^{pq} = -\epsilon^{[pqrs} \bar{\rho}_{rs} \gamma_5 \gamma_d - \delta^{[pq}_{rs} \epsilon^{]st} \bar{\rho}_{st} \gamma_5 \gamma_m, \quad (68) \]

Eqs (67,68) satisfy the general definition of rheonomic constraints. We can then write:

\[ R^{ab} = R_{cd}^{ab} \wedge V^c \wedge V^d + (-\epsilon^{abrs} \bar{\rho}_{rs} \gamma_5 \gamma_c - \delta^{[a}_{rs} \epsilon^{b]mnst} \bar{\rho}_{st} \gamma_5 \gamma_m) \psi \wedge V^c, \quad (69) \]

\[ R^a = R^a_{mn} V^m \wedge V^n, \quad (70) \]

\[ \rho = \rho_{ab} V^a \wedge V^b, \quad (71) \]

and we see that the equations of motion in superspace imply the rheonomic constraints:

\[ \epsilon \mid \rho = 0, \quad (72) \]
\[ \mathcal{E} | R_{ab} = 2 \left[ -\mathcal{E}^{ab\nu} \mathcal{P}_{\nu} \gamma_{5} \gamma_{c} - \delta^{[a}_{c} \mathcal{E}^{b]mst} \mathcal{P}_{m} \gamma_{5} \gamma_{n} \right] \mathcal{E} V^{c}, \quad (73) \]

\[ \mathcal{E} | R^{a} = 0, \quad (74) \]

The integrability conditions \((d^{2} = 0)\) are given by Bianchi identities:

\[ \mathcal{D} R_{ab} = 0, \quad (75) \]

\[ \mathcal{D} R^{a} + R_{ab} \wedge V^{b} - i \overline{\psi} \wedge \gamma^{a} \rho = 0, \quad (76) \]

\[ \mathcal{D} \rho + \frac{1}{4} R_{ab} \wedge \gamma_{ab} \psi = 0. \quad (77) \]

Inserting the rheonomic parametrization of curvatures in Bianchi identities, we get:

\[ \mathcal{D} \left( R_{mn} V^{m} \wedge V^{n} \right) + \mathcal{D} \left( \overline{\theta}_{ab} \psi \wedge V^{c} \right) = 0, \quad (78) \]

\[ R_{mn} \wedge V^{m} \wedge V^{n} \wedge V_{b} + \overline{\theta}_{ab} \psi \wedge V^{c} \wedge V_{b} - i \overline{\psi} \wedge \gamma^{a} \rho_{mn} V^{m} \wedge V^{n} = 0, \quad (79) \]

\[ \mathcal{D} \left( \rho_{ab} V^{a} \wedge V^{b} \right) + \frac{1}{4} \gamma_{ab} \psi \wedge \left( R_{mn} V^{m} \wedge V^{n} + \overline{\theta}_{ab} \psi \wedge V^{c} \right) = 0. \quad (80) \]

From the \(\psi \wedge V \wedge V\) projection of Eq. (79) we get the spacetime equation of gravitino:

\[ E_{p} = \gamma_{5} \gamma_{r} \mathcal{E}^{prst} \rho_{st} = \gamma_{5} \gamma_{r} \tilde{\rho}_{pr} = 0. \quad (81) \]

Eq. (78) gives the Einstein spacetime equation and the symmetry property of the Ricci tensor:

\[ \frac{3}{2} \left( R_{n}^{b} - \frac{1}{2} \delta_{n}^{b} R^{\infty} \right) = 0, \quad (82) \]

\[ R_{\alpha_{b}}^{\alpha_{b}} = R^{\alpha_{b}}_{\alpha_{b}}. \quad (83) \]

From Eqs (78-80) we get also both spacetime identities satisfied by the Riemann tensor:

\[ R_{[bimn]} = 0, \quad (84) \]

\[ \mathcal{D} [l_{b} R_{mn}^{ab}] = 0. \quad (85) \]

From Eqs (69-71), considering the expressions for \(\mathcal{E} | R^{ab}, \mathcal{E} | R^{a} \mathcal{E} | \rho\), and inserting them in the general expressions of Lie derivatives, we get:

\[ \delta_{\varepsilon} \omega^{ab} = (\nabla \varepsilon)^{ab} + 2 \varepsilon^{r} V^{r} R_{rs}^{ab} + 2 \overline{\theta}_{ab} \mathcal{E} \psi \mathcal{E} + 2 \overline{\theta}_{ab} \mathcal{E} V^{c}, \quad (86) \]
\[ \delta V^\alpha = (\nabla \varphi)^\alpha, \quad (87) \]
\[ \delta \varphi = \nabla \varphi + 2 \varphi \rho_n V^s. \quad (88) \]

2.6. Invariance of action and “off-shell” supersymmetry

The previous supersymmetry transformations are symmetries of the field equations; the action is not necessarily invariant with respect to them. The invariance of the action does not coincide with the invariance of the Lagrangian; even if \( \mathcal{L} \) remains invariant, the integration volume \( M_D \) changes under a fermionic diffeomorphism; \( M_D \) is in fact deformed by every infinitesimal transformation in directions \( \theta \). The bosonic coordinates transformations are always symmetries of the action, while the fermionic diffeomorphisms are symmetries of the action only if the action does not depend by the choice of the integration area \( M_D \). Using the Stokes theorem, we can write:

\[ A(M_D + \delta M_D) - A(M_D) = \int_{\Omega} d \mathcal{L}, \quad (89) \]

where \( M_D + \delta M_D \) is the new integration surface after the infinitesimal diffeomorphism and \( \Omega \) is the super-volume between the two surfaces (Figure 6). The diffeomorphisms in superspace are an “off-shell” invariance of the action if:

\[ d\mathcal{L} = 0, \quad (90) \]

i.e. if \( \mathcal{L} \) is a closed form in superspace.

![Figure 6. Application of Stokes theorem.](image)

Being Eq. (90) different from zero, the supersymmetry transformations are not an “off-shell” invariance of the action. A set of infinitesimal transformations constitutes an “invariance” if it keeps invariant the action; however, this set can not close an algebra “off-shell”. A set of transformations is however a “symmetry” if it keeps invariant the action and closes an algebra “off-shell”.

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In the spacetime approach the supersymmetry transformations close an algebra only on “on-shell”, not “off-shell” fields, however they are an invariance of the action. In the case of \( N = 1 \) supergravity we find that, explicitly performing the exterior derivative of the Lagrangian and contracting along the vector \( \mathcal{E} \), it is \( \mathcal{E} | d \mathcal{L} = 0 \) if and only if the rheonomic constraints are satisfied. Since these constraints imply the equations of motion, these transformations close an algebra only on “on-shell” configurations of the theory.

### 2.7. Construction of \( N = 1 \) supergravity theory using the rheonomic principle and Bianchi identities

There is another line of approach for the construction of a supergravity theory. It is possible to assume the rheonomic principle as fundamental starting point, in the usual form, i.e. the external components of the 2-forms of curvature in superspace have to be expresses “only” in terms of internal components. We write then the most general form of the curvatures in superspace according to the rheonomic principle. Obviously any parametrization of curvatures in superspace, in addition to satisfy the rheonomic principle, must satisfy the Lorentz covariance and a scaling invariance which we now define; this because the definition of curvatures and the corresponding Bianchi identities (69-71) and (75-77) are Lorentz-covariant and are also invariant under the following scaling transformation:

\[
\begin{align*}
\omega^{ab} &\to \omega^{ab} \quad (91); \\
R^{ab} &\to R^{ab} \quad (92) \\
V^a &\to wV^a \quad (93); \\
R^a &\to wR^a \quad (94) \\
\psi &\to \sqrt{w} \psi \quad (95); \\
\rho &\to \sqrt{w} \rho \quad (96)
\end{align*}
\]

with “\( w \)” constant parameter, different from zero, called “scaling factor”. Also the parametrization of curvatures in superspace must therefore respect the Lorentz-covariance and the scaling invariance (91-96). This, together with the rheonomic principle, allows to write in a non-complex way the general parameterization of curvatures in superspace.

In the case of \( N = 1 \) supergravity in absence of matter, the most general parametrization of curvatures in superspace is given by relations (61-63). Considering the rheonomic principle, tensors \( \Theta^{ab}_c \), \( K^{ab} \), \( H^F \), \( \Omega_{\alpha\beta} \) must be built with the “in” curvatures, i.e. in terms of spacetime curvatures \( R^{ab}_{mn} \), \( R^a_{mn} \), \( \rho_{mn} \).

With a re-definition of the spin connection, it is possible to put \( R^a=0 \), then \( R^a_{mn}=0 \). From relations (61-63) and (91-96) it follows that the scaling dimensions of the introduced tensors are:

\[
\begin{align*}
[\Theta^{ab}_c] = & -3/2 \quad (97); \\
[H^F] = & -1 \quad (98); \\
[K^{ab}] = & -1 \quad (99); \\
[\Omega_{\alpha\beta}] = & -1/2 \quad (100).
\end{align*}
\]

Introducing parametrizations (61-63) in the Bianchi identity of torsion and considering the \( \psi \wedge V \wedge V \) projection of Eq. (44) with \( R^a=0 \), we get:
\[ \bar{\theta}^{ab}_{c} \psi \wedge V_b \wedge V^c + i \overline{\psi} \wedge \gamma^a \rho_{bc} V^b \wedge V^c = 0, \quad (101) \]

or, equivalently:

\[ \frac{1}{2} (\bar{\theta}_{ab/c} - \bar{\theta}_{ac/b}) \psi = i \bar{\rho}_{bc} \gamma^a \psi. \quad (102) \]

Resolving it, we get:

\[ \bar{\theta}_{ab/c} = 2 i \bar{\rho}_{c[a} \gamma_{b]} - i \bar{\rho}_{ab} \gamma^c. \quad (103) \]

Then we fix the curvatures in superspace as follows:

\[ R^{ab} = R^{ab}_{cd} V^c \wedge V^d + (2 i \bar{\rho}_{c[a} \gamma_{b]} - i \bar{\rho}_{ab} \gamma^c) \psi \wedge V^c, \quad (104) \]

\[ R^a = 0, \quad (105) \]

\[ \rho = \rho_{ab} V^a \wedge V^b. \quad (106) \]

From Bianchi identities we can get also the spacetime equations of motion. Considering the \( \psi \wedge \psi \wedge V \) sector of gravitino Bianchi identity (77) and using Eqs (104-106), we get:

\[ \rho_{ab} \overline{\psi} \wedge \gamma^a \psi \wedge V^b + \frac{1}{4} \gamma^{ab} \psi \wedge \overline{\psi} (\gamma_c \rho_{ab} - 2 \gamma_a \rho_{bc}) \wedge V^c = 0. \quad (107) \]

Using the Fierz decomposition:

\[ \psi \wedge \overline{\psi} = \frac{1}{4} \gamma^a \psi \wedge \gamma_a \psi - \frac{1}{8} \gamma^{ab} \psi \wedge \gamma_{ab} \psi, \quad (108) \]

we obtain two equations for the coefficients of \( \psi \wedge \gamma_a \psi \wedge V^b \) and \( \psi \wedge \gamma_{pq} \psi \wedge V^c \) respectively:

\[ \frac{1}{8} \rho_{ab} + \frac{i}{16} \gamma^a \epsilon_{ab} \rho_{cd} + \frac{1}{16} \delta_{ab} \gamma^{pq} \rho_{pq} - \frac{1}{4} \gamma_{[a} \gamma^a \rho_{b]} = 0, \quad (109) \]

\[ -\gamma_{pq} \gamma^a \rho_{a}^c - 4 \gamma^{b} \gamma^c \rho_{pq} = 0. \quad (110) \]

This implies:

\[ \gamma_{s} \gamma_{r} \epsilon_{p}^{rst} \rho_{st} = \gamma^a \rho_{ap} = 0. \quad (111) \]

For getting the Einstein equation, we use Eq. (60) with \( R^a = 0 \) and differentiate it:

\[ \mathcal{D} (\gamma_s \gamma_{m} \rho \wedge V^m) = \gamma_s \gamma_{m} \mathcal{D} \rho \wedge V^m + \frac{i}{2} \gamma_s \gamma_{m} \rho \wedge \overline{\psi} \wedge \gamma^m \psi = 0. \quad (112) \]
Using the gravitino Bianchi identity, Eq. (112) becomes:

\[ \gamma_5 \gamma_m \left(-\frac{1}{4} R^{ab} \gamma_{ab} \wedge \psi \wedge V^m + \frac{i}{2} \rho \wedge \overline{\psi} \wedge \gamma^m \psi\right) = 0, \quad (113) \]

From \( \psi \wedge V \wedge V \wedge V \) sector, it is:

\[ \gamma_m \gamma_{ab} \psi \wedge R^{ab}_{\;\;cd} V^c \wedge V^d \wedge V^m = 0, \quad (114) \]

i.e.:

\[ i \gamma_5 \gamma^i \mathcal{E}_{mn} R^{ab}_{\;\;cd} V^c \wedge V^d \wedge V^m + 2 \gamma_b \gamma^i \mathcal{E}_{mn} R^{ab}_{\;\;cd} V^c \wedge V^d \wedge V^m = 0. \quad (115) \]

The second term of Eq. (115) is zero thanks to the cyclic identity of Riemann tensor, valid when \( R^a = 0 \). The first term gives the Einstein equation:

\[ R^m_{\;\;ab} - \frac{1}{2} \delta^m_a R^m_{\;\;mn} = 0, \quad (116) \]

having used: \( V^c \wedge V^d \wedge V^m = \epsilon^{cdm} \Omega_f \).

In conclusion, the rheonomic principle implemented in the Bianchi identities, with the Lorentz gauge invariance and the scaling property, have the same “on-shell” content of equations derived by the principle of extended action \([1,18-25]\).
Chapter III

KÄHLER MANIFOLDS

3.1. Introduction

The presence of states with spin 0 involves new structures (σ models) and new physical consequences (the super-Higgs phenomenon). The gauge fields $A^\mu$ become massive “eating” the degrees of freedom of a corresponding number of scalar fields. The spontaneous breaking of gauge symmetries through the Higgs mechanism is essential for the applications of the Yang-Mills theory to the description of interactions among elementary particles. It can thus glimpse the possibility of a spontaneous breaking of local supersymmetry, i.e. a super-Higgs phenomenon.

In theories which include scalar fields $\varphi$ and have a scalar potential $W(\varphi)$, we can have extremes $\varphi = \varphi_0$ that are not invariant under supersymmetry and therefore spontaneously break it. In this case, gravitinos corresponding to the generators of broken supersymmetry become massive “eating” the degrees of freedom of spin 1/2 fields, partners of scalar fields $\varphi$. This possibility is the basis of every phenomenological application of supergravity. For having indeed consistency with phenomenology, supersymmetry must occur in Nature as a spontaneously broken symmetry.

One of the main reasons for considering supergravity in the context of particle physics is its ability to solve, in the spontaneously broken version, the problem of the “gauge hierarchy”, i.e. to stabilize the ratio between the mass scale of weak interactions and that of grand unification:

$$\frac{M_W}{M_X} \cong 10^{-12}. \quad (117)$$

In Yang-Mills theory the Higgs potential $W(\varphi)$ results an arbitrary function, introduced “by hand” in the theory, on the contrary in supergravity models $W(\varphi)$ results from supersymmetry, is not “outside imposed”, is a consequence of the symmetry. Generally the same scalar field, that breaks supersymmetry, may also break bosonic gauge symmetries.

For understanding the characteristics of supergravity theories in relation to the sector of scalar fields, is interesting to discuss their formal structure in the presence of spin 0 fields. The scalar fields $\varphi^j$, regardless of the belonging multiplet, can be considered as the coordinates of a convenient differentiable manifold $\mathcal{M}$ with a Riemannian metric $g_{ij}(\varphi)$. The choice of the multiplet, of the number of supersymmetries $N$ and of the spacetime dimension $D$ is reflected in geometric properties of the manifold $\mathcal{M}$. The scalar manifolds $\mathcal{M}$ are normally Kähler manifolds. This fact has been seen in general, considering the constraints that supersymmetry imposes on supersymmetric Lagrangians involving Wess-Zumino multiplets. It has been found that the couplings described by these Lagrangians are compatible with supersymmetry only if the scalar fields $(0^+,0^-)$ parametrize a complex Riemannian manifold with a Kähler structure. In particular, the complex field:

$$z^i = A^i + i B^i, \quad (118)$$
3.2. Almost complex and complex structures on a 2n-dimensional manifold

Let $\mathcal{M}$ be a $2n$-dimensional manifold, with $T(\mathcal{M})$ as tangent space and $T^*(\mathcal{M})$ as cotangent space. The vectors of $T(\mathcal{M})$ are the linear differential operators:

$$p = t^\alpha(\varphi) \partial_\alpha = t^\alpha(\varphi) \frac{\partial}{\partial \varphi^\alpha}. \quad (119)$$

The vectors of $T^*(\mathcal{M})$ are the differential 1-forms $\omega = d\varphi^\alpha \omega_\alpha(\varphi)$. It is possible to consider linear operators $L$ on $T(\mathcal{M})$:

$$L : T(\mathcal{M}) \to T(\mathcal{M}) \quad (120)$$

such that:

$$\forall p' \in T(\mathcal{M}): \quad p'L \in T(\mathcal{M}); \quad (121)$$

$$\forall \alpha, \beta \in \mathbb{C}, \quad \forall p_1', p_2' \in T(\mathcal{M}): \quad (\alpha p_1' + \beta p_2')L = \alpha p_1'L + \beta p_2'L. \quad (122)$$

In every local chart, $L$ is represented by a mixed tensor $L^\alpha_\beta(\varphi)$:

$$p'L = t^\alpha(\varphi) L^\alpha_\beta(\varphi) \frac{\partial}{\partial \varphi^\beta}. \quad (123)$$

The action of $L$ is naturally translated on the cotangent space:

$$L : T^*(\mathcal{M}) \to T^*(\mathcal{M}) \quad (124)$$

$$L \omega = d\varphi^\alpha L^\alpha_\beta(\varphi) \omega_\beta(\varphi). \quad (125)$$

A $2n$-dimensional manifold $\mathcal{M}$ is called “almost complex” if it has an “almost complex structure”. An almost complex structure is a linear operator:

$$J : T(\mathcal{M}) \to T(\mathcal{M}) \quad (126)$$

that satisfies the property: $J^2 = -1$. In every local chart the operator $J$ is represented by a tensor $J^\alpha_\beta(\varphi)$ such that:

$$J^\alpha_\beta(\varphi) J^\beta_\gamma(\varphi) = -\delta^\gamma_\alpha. \quad (127)$$

Moreover, through a convenient basis change, in every point $P \in \mathcal{M}$ is possible to write $J^\alpha_\beta(\varphi)$ in the form:
A local frame with $J$ given by (128) is called “well-adapted frame”. Naming the basis vectors of the well-adapted frame, we have $\mathcal{P}_\alpha J = - \mathcal{P}_{\alpha+n}$; $\alpha \leq n$ and $\mathcal{P}_n J = \mathcal{P}_{n-1}$; $\alpha > n$. Introducing a Latin index $i = 1, \ldots, n$, we define the complex vectors $\mathcal{E}_i = \mathcal{P}_i - i \mathcal{P}_{i+n}$, $\mathcal{E}_{i\mu} = \mathcal{P}_i + i \mathcal{P}_{i+n}$, obtaining $\mathcal{E}_i J = i \mathcal{E}_i$, $\mathcal{E}_{i\mu} J = - i \mathcal{E}_{i\mu}$. The tangent vectors $\mathcal{E}_i$ are the partial derivatives along the complex coordinates $z^i = \varphi^i + i \varphi^{i+n}$, and $\mathcal{E}_{i\mu}$ are the partial derivatives along the complex conjugates $\bar{z}^\mu = \varphi^\mu - i \varphi^{\mu+n}$. The existence of the almost complex structure ensures that in any point $P \in \mathcal{M}$ it is possible to replace $2n$ real coordinates with $n$ complex coordinates, corresponding to a “well-adapted frame”. Furthermore each pair of “well-adapted frames” is linked to each other by means of a coordinate transformation that is a holomorphic function of the complex coordinates.

3.3. Hermitian and Kähler metrics

A metrics $g$ is a symmetric bilinear scalar-valued functional on $T(\mathcal{M}) \otimes T(\mathcal{M})$:

$$g : T(\mathcal{M}) \otimes T(\mathcal{M}) \rightarrow \mathbb{R}; \quad (129)$$

$$g(\mathcal{U}, \mathcal{W}) = g_{\alpha\beta} u^\alpha w^\beta. \quad (130)$$

It is $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$; $u^\alpha$ and $w^\beta$ are the components of tangent vectors $\mathcal{U}^\alpha$ and $\mathcal{W}^\alpha$. Let $\mathcal{M}$ a $2n$-dimensional manifold with an almost-complex structure $J$. A metrics $g$ on $\mathcal{M}$ is called “Hermitian with respect to $J$” if:

$$g(\mathcal{U}, \mathcal{W}) = g(\mathcal{V}, \mathcal{W}). \quad (131)$$

An almost-complex manifold with a Hermitian metrics $g$ is called “almost-complex Hermitian manifold”. $g(\mathcal{U}, \mathcal{W})$ can be written as follows:

$$g(\mathcal{U}, \mathcal{W}) = g_{\alpha\beta} u^\alpha w^\beta = g_{ij} u^i w^j + g_{i\mu} u^i w^\mu + g_{\mu j} u^\mu w^j + g_{\mu\nu} u^\mu w^\nu. \quad (132)$$

The following properties hold:

(i) reality of $g(\mathcal{U}, \mathcal{W})$: $g_{ij} = (g_{ij})^*$, $g_{i\mu} = (g_{i\mu})^*$;

(ii) symmetry of $g(\mathcal{U}, \mathcal{W})$: $g_{ij} = g_{ji}$, $g_{\mu j} = g_{j\mu}$.
(iii) Hermiticity of $g(\bar{u}^l, \bar{v}^k)$: $g_{ij} = g_{ji}^*$.

The metric $g$ is represented by a Hermitian matrix $g_{ij}^*$:

$$g_{ij}^* = (g_{ij})^* = g_{ji}^*.$$ (133)

The 2-form $\omega = i g_{ij}^* d\bar{z}^i \wedge dz^j$ is called “Kähler form”. A Hermitian metric on a complex manifold $\mathfrak{M}$ is called “Kähler metrics” if the associated 2-form is closed: $d\omega = 0$. A “Kähler manifold” is a complex Hermitian manifold with a Kähler metrics.

### 3.4. Differential geometry of Kähler manifolds

The general solution of every local chart of the differential equation $d\omega = 0$ is given by $g_{ij}^* = \partial_i \partial_j^* G$, with $G = G^* = G(z, \bar{z})$ real function of $z^i\ e\ \bar{z}^\mu$; $G$ is called “Kähler potential”.

Taking the convention that Greek indices embrace both $i$ and $i^*$, the Riemann affine connection associated with the Kähler metrics $g_{\alpha\beta} = g_{ij}^*$ has the form:

$$\left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\} = \frac{1}{2} g^{\alpha\mu} \left( \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\beta\gamma} \right);$$ (134)

with $g^{\alpha\mu} g_{\beta\mu} = \delta^\alpha_\beta$. It holds also:

$$\left\{ \begin{array}{c} i \\ j \end{array} \right\} = g^{\alpha\nu} \partial_\mu g_{\alpha\nu};$$ (135)

$$\left\{ \begin{array}{c} i \\ j \end{array} \right\} = \left\{ \begin{array}{c} i \\ j \end{array} \right\}^*;$$ (136)

$$\left\{ \begin{array}{c} i \\ j \end{array} \right\} = \left\{ \begin{array}{c} i \\ j \end{array} \right\} = 0;$$ (137)

$$\left\{ \begin{array}{c} i \\ j \end{array} \right\} = \left\{ \begin{array}{c} i \\ j \end{array} \right\} = 0.$$ (138)

The covariant differential of an object $v^i$ that transforms as a covariant vector is given by:

$$\nabla v^i = dv^i + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} d\bar{z}^j v^k.$$ (139)

The Kähler Riemannian curvature is given by:

$$\nabla^2 v^i = R^i_{j} v^j;$$ (140)

$$R^i_{j} = R_{m\nu}^{i} \left\{ \begin{array}{c} i \\ m \nu \end{array} \right\} d\bar{z}^m \wedge d\bar{z}^\nu;$$ (141)

$$R_{m\nu}^{i} = \partial_\nu \left\{ \begin{array}{c} i \\ m \end{array} \right\}.$$ (142)
The Ricci tensor is given by:

$$R_{mn} = R_{mi}^{\ i} = \partial_m \partial_n \ln \sqrt{g}; \quad (143)$$

$$g = \det [g_{\alpha\beta}] = \left( \det [g_{ij}] \right)^2, \quad (144)$$

having used the relation

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} = \partial_\beta \left( \ln \sqrt{g} \right) [1,26-29].$$
Chapter IV

\[ D = 4, \ N = 1 \] SUPERGRAVITY COUPLED TO \( n \) SCALAR MULTIPLETS

4.1. Kähler geometry for the coupling of scalar multiplets to \( D = 4, \ N = 1 \) supergravity

In \( \ N = 1 \) supergravity we have the vierbein \( V^a \), the gravitino \( \Psi \) and the spin connection \( \omega^{ab} \). From the particle viewpoint \( V^a, \Psi \) and \( \omega^{ab} \) describe the \( \ N = 1 \) gravitational multiplet \([2, 3/2]\). Now we couple such multiplet to \( n \) Wess-Zumino multiplets \([1/2, 0^+, 0^-]\), described by the set of 0-forms \([A^i, A^i', B^i]\), \((i = 1, \ldots\), \(n\)), with \( A^i \) and \( B^i \) a real scalar and a real pseudoscalar respectively, and \( A' \) a Majorana spinor. It is possible to introduce a set of complex fields \( z^i \):

\[
z^i = A^i + i B^i; \quad \bar{z}^i = (z^i)^* = A^i - i B^i, \quad (145)
\]

and we consider it as coordinates of a \( n \)-dimensional complex manifold \( \mathfrak{M} \) with a Kähler structure. On \( \mathfrak{M} \) the Kähler potential is introduced:

\[
G = G(\ z^i, \bar{z}^i); \quad G = G^*. \quad (146)
\]

We introduce also the chiral projections of spinors \( \Lambda^i \) and \( \Psi^i \):

\[
\Lambda^i = \chi^i + \chi'^i; \quad \chi^i = \frac{1 + \gamma^5}{2} \Lambda^i; \quad \chi'^i = \frac{1 - \gamma^5}{2} \Lambda^i; \quad (147)
\]

\[
\gamma_5 \chi^i = \chi^i; \quad \gamma_5 \chi'^i = -\chi'^i; \quad \chi'^i = C \gamma_0^T (\chi'^i)^*; \quad (148)
\]

\[
\Psi = \Psi^i + \Psi^i; \quad \Psi^i = \frac{1 + \gamma^5}{2} \Psi; \quad \Psi^i = \frac{1 - \gamma^5}{2} \Psi; \quad (149)
\]

\[
\gamma_5 \Psi^i = \Psi^i; \quad \gamma_5 \Psi^i = -\Psi^i; \quad \Psi^i = C \gamma_0^T (\Psi^i)^*; \quad (150)
\]

\[
\bar{\Psi}^i = (\Psi^i)^*, \quad \gamma_0 = \Psi^i (\frac{1 + \gamma^5}{2}) \gamma_0 = \bar{\Psi}^i (\frac{1 - \gamma^5}{2}); \quad (151)
\]

\[
\bar{\Psi} = (\Psi^i)^* \gamma_0 = \Psi^i (\frac{1 - \gamma^5}{2}) \gamma_0 = \bar{\Psi}^i (\frac{1 + \gamma^5}{2}); \quad (152)
\]

\[
\bar{\chi}'^i = (\chi'^i)^* \gamma_0 = \bar{\Lambda} \left( \frac{1 + \gamma^5}{2} \right); \quad (153)
\]

\[
\bar{\chi}'^i = (\chi'^i)^* \gamma_0 = \bar{\Lambda} \left( \frac{1 - \gamma^5}{2}. \quad (154)
\]
The coupling of scalar multiplets to supergravity corresponds to the construction of a “cross-section” of the bundle $B(\mathbb{R}^{4d}, \mathfrak{K})$, that has the $N=1$ superspace $\mathbb{R}^{4d}$ as support space and the Kähler manifold $\mathfrak{K}$ as bundle. The coordinate $z$ is a superfield $z'=z(x, \theta)$, then at every point $(x, \theta) \in \mathbb{R}^{4d}$ of the support it is associated a point $z' \in \mathfrak{K}$ of the bundle. Expanding $dz'$ in the basis $(V, \psi)$, we find:

$$
\begin{align*}
 dz' &= Z'_{a} V^{a} + \overline{\psi}' \psi' = Z'_{a} V^{a} + \overline{\psi}' \chi', \\
 d\overline{z}' &= \overline{Z}'_{a} V^{a} + \overline{\psi}' \psi' = \overline{Z}'_{a} V^{a} + \overline{\psi}' \chi',
\end{align*}
$$

(155)  (156)

considering that the “out” component of $dz$ is $\chi$, field of spin 1/2. $Z'_{a}$ is a vector field and $\overline{\chi}'$ a left-handed spinor field. The Kähler connection is defined as:

$$
Q = \frac{1}{2i} (\partial_{j} G dz' - \partial_{j} G d\overline{z}').
$$

(157)

The curvature of the Kähler connection is the 2-form $K$ defined as:

$$
K = dQ = i g_{ab} dz' \wedge d\overline{z}'^a.
$$

(158)

Using relations (155, 156) and defining the quantities:

$$
K_{ab} = i g_{ab} Z^{i}_{[a} \overline{Z}^{j}_{b]}, \\
T_{a} = \overline{\chi}' \gamma_{a} \chi^{j}, \\
\Sigma^{a}_{.} = i g_{aj} \chi^{j} \overline{Z}^{i}_{a}, \\
\Sigma^{.}_{a} = (\Sigma^{a}_{.})^{j} = -i g_{ij} \chi^{j} Z^{i}_{a},
$$

(159)  (160)  (161)  (162)

it is possible to write:

$$
K = K_{ab} \wedge V^{a} \wedge V^{b} + \frac{i}{2} T_{a} \overline{\psi} \wedge \gamma^{a} \psi' + \overline{\psi} \Sigma^{a}_{.} \wedge V^{a} + \overline{\psi}' \Sigma^{.}_{a} \wedge V^{a}.
$$

(163)

The outer derivatives of the matter fields $z'$ and $\chi'$ are the analogous of curvatures $R^{ab}$, $R^{a}$ and $\rho$ of the supergravity fields. It is convenient to define as “curvature” of $\chi$ the covariant derivative $\nabla \chi'$, that is covariant with respect to Lorentz transformations, Kähler transformations, coordinates transformations on Kähler manifold.

The set of curvatures of supergravity and Wess-Zumino multiplets is therefore:

$$
R^{a} = \mathcal{D} V^{a} - i \overline{\psi} \wedge \gamma^{a} \psi' , \quad (\mathcal{D} V^{a} \equiv dV^{a} - \omega^{ab} \wedge V^{b}).
$$

(164)
\[ R^{ab} = d \omega^{ab} - \omega^a_c \wedge \omega^b_c \equiv \mathcal{R}^{ab}, \quad (165) \]
\[ \rho_\star = \nabla \psi_\star, \quad (166) \]
\[ R(z)^{\text{def}} = dz^i = dz^i, \quad (167) \]
\[ R(\chi)^{\text{def}} = \nabla \chi^i = \nabla \chi^i, \quad (168) \]

with:

\[ \nabla \psi_\star = d \psi_\star - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab} \psi_\star + \frac{i}{2} Q \wedge \psi_\star; \quad (169) \]
\[ \nabla \chi^i = d \chi^i - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab} \chi^i + \left\{ i \frac{1}{j k} \right\} dz^j \chi^k + \frac{i}{2} Q \chi^i; \quad (170) \]

The Bianchi identities of supergravity + Wess-Zumino multiplets are given by:

\[ \mathcal{D} R^a + R^{ab} \wedge V_b + i \bar{\psi}^a \wedge \gamma^a \psi_\star - i \psi^a \wedge \gamma^a \rho_\star = 0; \quad (171) \]
\[ \mathcal{D} R^{ab} = 0; \quad (172) \]
\[ \nabla \rho_\star + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi_\star - \frac{i}{2} K \wedge \psi_\star = 0; \quad (173) \]
\[ d d z^i = 0; \quad (174) \]
\[ \nabla \nabla \chi^i + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \chi^i + i \frac{1}{2} K \chi^i - R_{m}^{a} j \ i \ dz^{m} \wedge d z^{i} \chi^{j} = 0. \quad (175) \]

### 4.2. Solutions of Bianchi identities and auxiliary fields

In Bianchi identities (171-175) we insert the equation of the torsion \( R^a = 0 \) and the rheonomic condition \( dz^i = Z^i_a V^a + \chi^i \psi_\star \). The first condition is a kinematic constraint that can always be imposed by re-defining the spin connection, while the rheonomic condition defines who is the supersymmetric partner of \( z^i \). To give these two conditions is equivalent to fix the supersymmetry transformations. They fix the rules of supersymmetry transformation of \( V^a \) and \( z^i \):

\[ \delta_\epsilon z^i = \chi^i \epsilon_\star, \quad (176) \]
\[ \delta_\epsilon V^a = i \epsilon^a \gamma^a \psi_\star - i \bar{\psi}^a \gamma^a \epsilon_\star, \quad (177) \]
\[ Z^{i_a} V^a_{\mu} \] is not equal to \( \delta_{\mu} z^i \), as would happen if the theory was formulated only on spacetime.

For finding the solution of Bianchi identities (171-175) we insert the most general rheonomic parameterization of curvatures \( R^{ab} \) and \( \rho \), given by:

\[
R^{ab} = R^c_{\phantom{c}cd} V^c \wedge V^d + \tilde{\Omega}^{ab}_{\phantom{ab}c} \psi \wedge V^c + \bar{\psi} \wedge K^{ab} \psi, \tag{178}
\]

\[
\rho = \rho^{ab}_{\phantom{ab}a} V^a \wedge V^b + H_c \psi \wedge V^c + \Omega_{ab} \psi^a \wedge \psi^b, \tag{179}
\]

In the case of non-coupling to matter supergravity, the values of \( \tilde{\Omega}^{ab}_{\phantom{ab}c} \), \( K^{ab} \), \( H_c \), \( \Omega_{ab} \) have been fixed in (67). In the presence of matter fields, the values of these tensors can also depend on the matter fields, because we can use for their construction also the “inner” fields \( Z^{i_a} e \nabla_\mu Z^i \). Resolving the Bianchi identities of the “off-shell” supergravitational multiplet and introducing the new general parameterizations in the complete Bianchi identities of supergravity + Wess-Zumino multiplets (171-175), we obtain the explicit form of \( \tilde{\Omega}^{ab}_{\phantom{ab}c} \), \( K^{ab} \), \( H_c \), \( \Omega_{ab} \) in terms of matter fields and supergravity fields.

The \( \psi \wedge \psi \wedge V \) sector brings to an equation that breaks in two parts:

\[
\bar{\psi} \wedge \gamma^{[a} H^{b]} \psi = 0, \tag{180}
\]

\[
\bar{\psi} \wedge \gamma^{[a} H^{b]} \psi = -i \bar{\psi} \wedge K^{ab} \psi. \tag{181}
\]

Eq. (180) is resolvable decomposing \( H^b \) in a Dirac complete basis; the general solution is:

\[
H_m = i A_m \gamma_5 + \phi \gamma_5 \gamma_m + i \delta \gamma_m + i \gamma_5 \gamma_m^u A'_u, \tag{182}
\]

with \( A_m \) and \( A'_m \) two axial vectors, \( \delta \) and \( \phi \) a scalar and a pseudoscalar respectively. Substituting Eq. (182) in (181), we get:

\[
K^{ab} = -i \phi \gamma_5 \gamma^{ab} - \delta \gamma^{ab} - i \varepsilon^{abcd} A'_c \gamma_d. \tag{183}
\]

The \( \psi \wedge \psi \wedge \psi \) sector of (171) brings to \( -i \bar{\psi} \wedge \gamma^a \Omega = 0 \Leftrightarrow \bar{\Omega} \wedge \gamma^a \psi = 0 \), with \( \Omega = \Omega_{ab} \psi^a \wedge \psi^b \). Writing in complete generality \( \Omega = i \zeta_a \bar{\psi} \wedge \gamma^a \psi - \frac{1}{2} \bar{\xi}_{ab} \bar{\psi} \wedge \gamma^{ab} \psi \) and using the decomposition in irreducible representations \( \zeta_a = \xi^{(12)}_a + \frac{1}{4} \gamma^a \xi^{(4)} \) and \( \xi_{ab} = \xi^{(8)}_{ab} - \gamma_{[a} \xi^{(12)}_{b] - \frac{1}{12} \gamma_{ab} \xi^{(4)} \), we get:

\[
\Omega = -\frac{1}{2} \gamma^a \zeta \bar{\psi} \wedge \gamma^a \psi + \frac{1}{2} \gamma_{ab} \zeta \bar{\psi} \wedge \gamma^{ab} \psi, \tag{184}
\]
with $\zeta$ Majorana spinor. Introducing the complex field $S = \phi + i \bar{\phi}$ and using the chiral notation, we obtain:

$$R^a = 0; \quad (185)$$

$$\rho_* = \rho_+^a V_a \wedge V_b + i A_a \psi_+ \wedge V^a + i A^*_a \gamma^{ab} \psi_+ \wedge V_b + S \gamma_+ \psi^+ \wedge V_a + \psi_+ \wedge \overline{\psi}_* - \psi_+ \wedge \overline{\psi}^* \zeta^*; \quad (186)$$

$$R^{ab} = R^{ab}_{cd} V^c \wedge V^d - (2i \overline{\psi}_+ \gamma^{[a} \rho_+ b]c + 2i \overline{\psi}_+ \gamma^{[a} \rho b]c - i \overline{\psi}_+ \gamma^c \rho_+ a b -$$

$$-i \overline{\psi}_+ \gamma^c \rho_+ a b \psi_+ \wedge V_c - i S \overline{\psi}_+ \gamma^{ab} \psi_+ \wedge V_a + i S \overline{\psi}^+ \gamma^{ab} \psi^+ - 2i A^*_c \overline{\psi}_+ \gamma^{cd} \psi_+ \epsilon^{abcd}. \quad (187)$$

The case of pure supergravity is obtainable putting:

$$A_a = A'_a = S = \zeta_1 = 0 \Rightarrow \zeta^* = 0. \quad (188)$$

It follows that in the theory coupled to matter the auxiliary fields $A_a, A'_a, S$ and $\zeta_1$ are identified with convenient functions of the matter fields $z^i$ and $\chi^i$. The scaling transformations are substantially extendable to the matter fields by placing:

$$z' \rightarrow z', \quad (189) \quad \chi^i \rightarrow \frac{1}{\sqrt{\lambda}} \chi^i, \quad (190)$$

with $\lambda$ real constant parameter. If we call $w(\phi)$ the scaling power of every field $\phi$, we have by rheonomic conditions:

$$w(V^a) = 1, \quad w(\psi) = 1/2, \quad w(\omega^{ab}) = 0, \quad (191)$$

and being the coordinates of the Riemannian manifold of scalars $z'$. scaling invariant, we get:

$$w(z') = 0 \Leftrightarrow w(Z'_{a}) = -1; \quad w(\chi') = -1/2; \quad w(A_a) = w(A'_a) = -1; \quad w(S) = -1; \quad w(\zeta) = -1/2. \quad (192)$$

The solution of Bianchi identities is completed analyzing the Bianchi identity of gravitino. We preliminarily observe that in (186) we can put $\zeta_1 = \zeta^* = 0$, because in absence of matter the gravitino curvature $\rho_*$ contains the Kähler connection $Q$ that was not entering in the definition of $\rho_*$ in absence of matter. The additional term $i/2 Q \wedge \psi_+$ is a 2-form whose component along $\psi^+ \wedge \psi_+$ can be identified with $\zeta$. Therefore we make no restrictions in placing $\zeta = 0$. We rewrite (186) in the form:

$$\rho_* = \rho_+^a V_a \wedge V_b + i A_a \psi_+ \wedge V^a + i A^*_a \gamma^{ab} \psi_+ \wedge V_b + S \gamma_+ \psi^+ \wedge V_a; \quad (193)$$
The corresponding Bianchi identity is:

$$\nabla \psi + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi + \frac{1}{2} g_{j*} dz^j \wedge d\bar{z}^* \wedge \psi = 0. \quad (194)$$

Introducing in the Bianchi identity of gravitino the parametrization (193) and that of $R^{ab}$ (187), in $\psi \wedge \psi \wedge \psi$ sector we find:

$$A_a \psi \wedge \bar{\psi} \wedge \gamma^a \psi^* + A_{a*} \gamma^{ab} \psi \wedge \gamma_{a} \psi^* + i S \gamma^a \psi^* \wedge \bar{\psi} \wedge \gamma^a \psi^* +$$

$$+ \frac{1}{4} \gamma_{ab} \psi \wedge (i S \bar{\psi} \wedge \gamma^{ab} \psi^* + i S \bar{\psi} \wedge \gamma^{ab} \psi^* + 2 i A_{c} \bar{\psi}^* \wedge \gamma_{c} \psi \epsilon^{abcd}) +$$

$$+ \frac{1}{2} g_{ij*} \psi \wedge [(\bar{\psi}^i \psi^*) \wedge (\bar{\psi}^j \psi^*)] = 0. \quad (195)$$

Considering that $\gamma_{ab} \psi \wedge \bar{\psi} \wedge \gamma^{ab} \psi^* = 0$, $\gamma^a \psi \wedge \bar{\psi} \wedge \gamma^a \psi^* = 0$ (Fierz), and $\gamma_{ab} \psi \wedge \bar{\psi} \wedge \gamma_{ab} \psi^* = 0$ (self-duality), we note that the $S$ field does not contribute. Being $\psi \wedge \bar{\psi} \wedge \gamma_{a} \psi^* = \Xi_{a*}$, $2 \psi \wedge \bar{\psi} \wedge \gamma_{a} \psi^* = \Xi_{a*}$, $\gamma_{ab} \psi \wedge \bar{\psi} \wedge \gamma_{b} \psi^* = - \Xi_{ab}$, $2 \gamma_{ab} \psi \wedge \bar{\psi} \wedge \gamma_{b} \psi^* = - \Xi_{ab}$, by (195) we have:

$$\frac{1}{2} A_a \Xi_{a}^{(12)} - \frac{1}{2} A_{a*} \Xi_{a}^{(12)} - \frac{1}{2} A_{a} \Xi_{a}^{(12)} + \frac{1}{4} g_{ij*} \psi \wedge \bar{\psi} \wedge \gamma_{a} \psi^* \gamma^i \chi^j = 0. \quad (196)$$

Then

$$\frac{1}{2} A_a - A_{a*} \Xi_{a}^{(12)} + \frac{1}{8} \Xi_{a}^{(12)} (g_{ij*} \bar{\psi} \gamma_{a} \chi^j) = 0, \quad \text{from which: } A_{a*} = \frac{1}{2} A_a +$$

$$+ \frac{1}{8} g_{ij*} \bar{\psi} \gamma_{a} \chi^j.$$

This last expression is the only object with correct scaling weight at which the fields $A_a$ and $A_{a*}$ can be identified; they are therefore not independent and in particular we choose $A_a = 0$.

We now consider the $V \wedge \psi \wedge \psi$ sector of the same equation; canceling the coefficient of the $\bar{\psi}^* \wedge \gamma^{lm} \psi^*$ current, we get $\nabla_{m} \bar{S} = 0$, with $\nabla_{m} = \partial_{m} - \frac{p^i}{2} G_{m}$. (it is $G_{m} \equiv \partial_{m} G$.) In our case it is $\nabla_{m} \bar{S} = \partial_{m} \bar{S} - \frac{1}{2} G_{m} \bar{S} = 0$, ($p^i = 1$), and this is equivalent to:

$$\nabla_{m} \bar{S} = e^{G/2} \partial_{m} (e^{-G/2} \bar{S}) = 0.$$ This implies $\partial_{m} (e^{-G/2} \bar{S}) = 0 \Rightarrow e^{-G/2} \bar{S} = f(z)$, where $f(z)$ is an arbitrary analytic function. So it is $\bar{S} = f(z) e^{G/2}$. On the other hand, the $S$ field must be pure imaginary (for the Majorana conditions), then the analytic function $f(z)$ must be equal to “i” times a constant indicated with “e”. The final solution is therefore:

$$S = i e e^{G/2}. \quad (197)$$

In the $\bar{\psi} \wedge \gamma^{lm} \psi$ sector we find an equation that, with multiplication of its members by $\gamma^{lm}$ and considering the algebra of $\gamma$-matrices, gives:
\[
\frac{i}{2} \gamma^a (\nabla_{(0,1)} A_a') - \frac{1}{8} \gamma^a \Theta^{ac'} b + \frac{3}{4} g_{ij} \chi' Z' b = 0. \quad (198)
\]

In the sector of one-index current of Eq. (194) we find an equation that, with multiplication by \( \gamma^a \), brings to the motion equation of gravitino; it will be given in a complete way in the next chapter with the coupling with the vector multiplets too.

The Bianchi identity (174) associated to the scalar field \( z^i \) can be used for the determination of parametrization of the covariant derivative \( \nabla \chi^i \). We write, in full generality:

\[
\nabla \chi^i = \nabla_{\alpha} \chi^i V^\alpha + \{ \chi^i \}_{\alpha} \gamma^\alpha \psi^* + \{ \chi^i \}_{ab} \gamma^{ab} \psi^* + \{ \chi^i \} \psi^*, \quad (199)
\]

where the coefficients \( \{ \chi \} \) depend by the field. Considering the outer derivative of the rheonomic parametrization (155) and inserting Eq. (199), from the \( \psi^* \psi^* \) projection we get an equation bringing to: \( \{ \chi \}_{\alpha} = i Z^\alpha \); \( \{ \chi \}_{ab} = 0 \); \( \{ \chi \} \) free.

The free function \( \{ \chi \} \) is the auxiliary field of the scalar multiplet; its Kähler weight is \( p = 1/2 \). The \( \psi^* \psi^* \) sector of the Bianchi identity (175) links \( \{ \chi \} \), assumed as function of the scalar fields \( (z^i, \bar{z}^i) \) to the auxiliary field of supergravity \( S \).

From the \( \psi^* \psi^* \) sector of the covariant derivative of \( \{ \chi \} \) we find \( \nabla \{ \chi \} = 2i \delta^i_m S^*, \) with:

\[
\nabla \{ \chi \} = \partial_m \{ \chi \} + \left\{ \begin{array}{l} i \\ m \\ j \end{array} \right\} \{ \chi \} j - \frac{1}{2} \partial_m G \{ \chi \}. \quad (200)
\]

This differential equation is resolvable through the ansatz \( \{ \chi \} = a g^{ij} \partial_j e^{bG} \), with \( g^{ij} g_{ij} = \delta^m_m \). Inserting it in (200) and considering (197), the equation is satisfied by:

\[
a = \frac{2e}{p} = 4e; \quad b = \frac{1}{2}. \quad (201)
\]

It has been therefore determined, only by the analysis of the Bianchi identity, the dependence of the gravitino field and of the auxiliary field of spin 1/2 from Kähler potential \( G(z, \bar{z}) \):

\[
S = ie^{\exp(G/2)}; \quad (202) \quad \{ \chi \} = 2e(g^{ij} \partial_j G) \exp(G/2). \quad (203)
\]

4.3. The construction of the action

As in pure supergravity, the action principle for the interacting system is given by

\[
A = \int_{M_4 \subset R^{11}} d\varphi, \quad \text{with} \quad \varphi \quad \text{a 4-form built with the basis fields of the theory. Making} \ A \ \text{stationary with respect both to variations in the fields and in the surface} \ M_4, \ \text{we obtain equations of differential forms which must be consistent with the rheonomic parametrizations already determined through the Bianchi identities.}
\]

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The rules for writing the ansatz for $\mathcal{L}$ are:

i) $\mathcal{L}$ must be built using only the outer product “$\wedge$” and the outer derivative “$d$”. It is excluded the Hodge duality and this is substituted by $\varepsilon_{abcd}$;

ii) $\mathcal{L}$ must be Lorentz invariant;

iii) $\mathcal{L}$ must be invariant with respect to general transformations of holomorphic coordinates in the Kähler manifold;

iv) $\mathcal{L}$ must respect the scaling invariance of $V^a$, $\omega^{ab}$, $\Psi$, $z^i$, $\chi^i$; all terms must have the same scaling weight of the Einstein term $w=2$;

v) $\mathcal{L}$ must contain the kinetic terms of all physical fields;

The most general Lagrangian that meets these requirements can be written as a sum of various sub-Lagrangians. First of all we divide $\mathcal{L}$ in a part $\mathcal{L}_1$ which survive to the limit “$e \to 0$” and a part $\Delta \mathcal{L}$ proportional to “$e$”: $\mathcal{L} = \mathcal{L}_1 + \Delta \mathcal{L}$. Then we divide $\mathcal{L}_1$ as follows:

$$\mathcal{L}_1 = \mathcal{L}_{(\text{KIN})} + \mathcal{L}_{(\text{PAULI})} + \mathcal{L}_{(\text{TORSION})} + \mathcal{L}_{(\text{4-FERMI})}(2\Psi V) + \mathcal{L}_{(\text{4-FERMI})}(4V), \quad (204)$$

with:

(a) $\mathcal{L}_{(\text{KIN})}$ contains the kinetic terms of all fields, in particular that of the scalar field, written in the first order formalism;

(b) $\mathcal{L}_{(\text{PAULI})}$ contains the terms that couple the bosonic derivative $dz^i$ to fermionic currents $\chi^i \gamma \Psi$;

(c) $\mathcal{L}_{(\text{TORSION})}$ contains the terms $R^a \wedge \ldots$ such that the variation $\delta \omega^{ab}$ produces the field equation $R^c = 0$;

(d) $\mathcal{L}_{(\text{4-FERMI})}(2\Psi V)$ contains the 4-Fermi not derivative terms of type $\chi \chi \gamma \Psi \wedge V \wedge V$;

(e) $\mathcal{L}_{(\text{4-FERMI})}(4V)$ contains the 4-Fermi not derivative terms of type $\chi \chi \chi \chi V \wedge V \wedge V \wedge V$. This part of the Lagrangian must be fixed by the supersymmetry invariance, that is $\mathcal{E} \lvert d \mathcal{L} = 0$.

We divide also $\Delta \mathcal{L}$ in the following way:

$$\Delta \mathcal{L} = \Delta \mathcal{L}_{(\Psi \Psi V V)} + \Delta \mathcal{L}_{(\Theta \Psi \Psi V V)} + \Delta \mathcal{L}_{(\Xi \Psi V V V)} + \Delta \mathcal{L}_{(\text{Potential})}, \quad (205)$$

with:

(f) $\Delta \mathcal{L}_{(\Psi \Psi V V)}$ is the mass term of gravitino, whose coefficient will be tied to the auxiliary field $S$;
(g) $\Delta \mathcal{L}_{(2VVVV)}$ is the “not diagonal” mass term of spin 1/2, spin 3/2, whose coefficient will be tied to the auxiliary field $H$;

(h) $\Delta \mathcal{L}_{(2VWWW)}$ is the mass term of spin 1/2, whose coefficient will be tied to the derivative of the next term;

(i) $\Delta \mathcal{L}_{(Potential)}$ is the potential term of the scalar field, that will be expressed as quadratic form in $S$ and $H$.

4.4. Construction of $\mathcal{L}_1$

For getting $\mathcal{L}_1$ we consider all terms, except $\mathcal{L}_{(4-\text{FERMI})}$, which is fixed at the end by a supersymmetry transformation. Let’s start with the following general ansatz:

$$\mathcal{L}_{(KIN)} = \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4(\overline{\psi}^* \gamma_a \rho + \overline{\rho}^* \gamma_a \psi^*) \wedge V^a +$$

$$+ i \delta_1 g_{ij} (\overline{X}^i \gamma_a \nabla \chi^j + \overline{\chi}^j \gamma_a \nabla \chi^i) \wedge V_b \wedge V_c \wedge V_d e^{abcd} +$$

$$+ \delta_2 g_{ij} Z^i_a (dz^i - \overline{\chi}^i \psi^*) \wedge V_b \wedge V_c \wedge V_d e^{abcd} +$$

$$+ \delta_3 g_{ij} Z^i_a (dz^i - \overline{\chi}^i \psi^*) \wedge V_b \wedge V_c \wedge V_d e^{abcd} +$$

$$+ \delta_4 g_{ij} Z^i_a Z^j_a e^{b,c,h} V^h \wedge V^b \wedge V^c \wedge V^h ; \ (206)$$

$$\mathcal{L}_{(PAULI)} = i \delta_5 g_{ij} \overline{\chi}^j \psi^a \wedge V^a \wedge V^b +$$

$$+ i \delta_6 g_{ij} \overline{\chi}^j \psi^a \wedge V^a \wedge V^b ; \ (207)$$

$$\mathcal{L}_{(TORSION)} = \delta_7 R^a \wedge V_a \wedge g_{ij} \overline{X}^i \gamma_b \chi^j \wedge V^b ; \ (208)$$

$$\mathcal{L}_{(4-\text{FERMI})_{(2V2V)}} = i \delta_8 g_{ij} \overline{X}^i \gamma_a \chi^j \wedge \overline{\psi}^* \gamma_b \psi^a \wedge V^a \wedge V^b . \ (209)$$

The first two terms correspond to the action of pure supergravity; the coefficients $\delta_1, \ldots, \delta_8$ are real and we can determine them by the motion equations. The sign of $\delta_1$ is fixed by requirement of positive energy: $\delta_1 = -\alpha$; ($\alpha > 0$). The Hermiticity of Lagrangian brings to: $\delta_2 = \delta_2$; $\delta_6 = -\delta_6$. By the variation in $\delta z^i_a$ we get: $\delta_4 = -\frac{1}{4} \delta_2 = -\frac{1}{4} \delta_3$.

By the variation in $\delta \chi^i$ we get a field equation from whose projections $\psi \wedge V \wedge V$ and $\overline{\psi} \wedge \psi \wedge V \wedge V$ it is: $\delta_2 = 2\alpha$; $\delta_3 = -6\alpha$; $\delta_8 = 6\alpha$. The variation in $\delta \omega^{ab}$ gives: $\delta_5 = -3\alpha$. Therefore we have:
\[\delta_{1} = -\alpha; \ \delta_{2} = 2\alpha; \ \delta_{3} = 2\alpha; \ \delta_{4} = -\frac{1}{2}\alpha; \ \delta_{5} = -6\alpha; \ \delta_{6} = 6\alpha; \ \delta_{7} = -3\alpha; \ \delta_{8} = 6\alpha. \quad (210)\]

Considering now the equation of gravitino (variation in \(\delta \psi^{*}\)) and setting \(A_{a} = \mu T_{a},\)
\(A'_{a} = \frac{V}{2} T_{a},\) with \(T_{a}\) given by (160), \(\mu\) and \(V\) coefficients to get, the projection \(\psi \wedge V \wedge V\)
gives: \(V = \frac{3}{2}\alpha, \ \mu = 0.\) This is consistent with what we have imposed on Bianchi identities. By
the Bianchi identity (194), with previous positions for \(A_{a}, A'_{a}\) and (202) we get, in the
projection \(\psi \wedge \psi \wedge \psi: \ V = \mu + \frac{1}{2} p = \frac{1}{4}.\) We can then rewrite all coefficients in terms of Kähler
charge of gravitational \(p = 1/2:\)

\[\delta_{1} = -\frac{1}{3}; \ \delta_{2} = \frac{2}{3}; \ \delta_{3} = \frac{2}{3}; \ \delta_{4} = -\frac{1}{6}; \ \delta_{5} = -2; \]
\[\delta_{6} = 2; \ \delta_{7} = -1; \ \delta_{8} = 2; \ \mu = 0; \ \nu = \frac{1}{4}. \quad (211)\]

About the calculation of \(\mathcal{L}^{(4-\text{FERMI})(4V)}\) the ansatz is given by:

\[\mathcal{L}^{(4-\text{FERMI})(4V)} = \epsilon_{abcd} V^{a} \wedge V^{b} \wedge V^{c} \wedge V^{d} \bar{\chi}^{i} \gamma_{m} \chi^{j} \bar{\chi}^{k} \gamma^{m} \chi^{l} \times\]
\[\times (m g^{ij} g_{kl} + n R_{ijkl}) , \quad (212)\]

and the values of \(m\) and \(n\) turn out to be: \(m = -\frac{p^{2}}{12} = -\frac{1}{48}; \ \ n = -\frac{p}{24} = -\frac{1}{48}.\) With these ones
the Lagrangian \(\mathcal{L}_{1}\) is totally built.

4.5. Construction of \(\Delta \mathcal{L}\)

By the equation of gravitino we see that if “\(e \neq 0\)”, the term \(-8 \gamma_{a} \rho_{c} \wedge V^{a}\) generates a
further term: \(8 S \gamma_{ab} \psi^{*} \wedge V^{a} \wedge V^{b}.\) To compensate it, the following term is added to the
Lagrangian:

\[\Delta \mathcal{L}_{(\psi \psi \psi \psi)} = -4 (S \bar{\psi}^{*} \wedge \gamma_{ab} \psi^{*} + S^{*} \bar{\psi}^{*} \wedge \gamma_{ab} \psi^{*}) \wedge V^{a} \wedge V^{b}. \quad (213)\]

By doing in this way, considering the equation corresponding to the variation in \(\delta \chi^{i}\),
for \(e \neq 0\) it generates an unbalanced term that can be deleted by adding to the Lagrangian the term:

\[\Delta \mathcal{L}_{(\chi^{i} \psi \psi \psi)} = (g^{i} \bar{\chi}^{i} \gamma_{a} \psi^{*} + g_{i} \bar{\chi}^{i} \gamma_{a} \psi^{*}) V^{a} \wedge V^{c} \wedge V^{d} \epsilon^{abcd}, \quad (214)\]
with: \( \mathcal{G}_i = -2i \delta g_\gamma^i \chi^j \); \( \mathcal{G}^*_i = (\mathcal{G}_i)^* \). Having introduced these terms, we must also introduce a term of potential:

\[
\Delta \mathcal{L}_{(\text{Potential})} = -W \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d , \tag{215}
\]

with: \( W = -2 S S^* - \frac{1}{2} \delta g_\gamma^i \chi^j \). Then we have:

\[
\Delta \mathcal{L}_{(\text{VVVV})} = (m_{ij} \chi^j + m_{i,j} \chi^j \chi^j) \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d . \tag{216}
\]

By the variation in \( \delta \chi^j \) of (216), by that in \( \delta \psi^* \) of (214) and that in \( \delta z^i \) of (215), we obtain the condition: \( \partial_i W - 2 m_{ij} \chi^j + \mathcal{G}_i S^* = 0 \). Considering ultimately that:

\[
S = i e \exp \left[ \frac{1}{2} G(z, \bar{z}) \right], \tag{217}
\]

\[
\chi^i = 2 e (g_\gamma^i \partial_j G) \exp \left[ \frac{1}{2} G(z, \bar{z}) \right], \tag{218}
\]

\[
W = -\frac{2}{3} e^2 (3 - g_\gamma^i \partial_i G \partial_j G) \exp \left[ \frac{1}{2} G(z, \bar{z}) \right], \tag{219}
\]

\[
\mathcal{G}_i = \frac{4}{3} i e \partial_i G \exp \left[ \frac{1}{2} G(z, \bar{z}) \right], \tag{220}
\]

these relations allow to obtain [1,30-34]:

\[
m_j = \frac{e}{6} (\partial_i G \partial_j G + \nabla_i \partial_j G) \exp \left[ \frac{1}{2} G(z, \bar{z}) \right]. \tag{221}
\]
Chapter V

COUPLING WITH VECTOR MULTIPLETS

5.1. Introduction

Scalar multiplets contain quarks, leptons and Higgs particles together with their superpartner, i.e. squarks, sleptons, Higgsins.

Gauge bosons, on the other hand, belong to the vector multiplets (1, 1/2). Similarly to Yang-Mills ordinary theory, the role of vector multiplets is to “make local” some groups of global symmetries of the matter's Lagrangian. The previously built Lagrangian has local supersymmetry, but it allows at most a group of global bosonic symmetries.

Such symmetries are in bijective correspondence with the isometries of Kähler metrics \( g_{ij}^y(z, \overline{z}) \) satisfying the further requirement to maintain invariant the Kähler potential \( G(z, \overline{z}) \).

If in fact \( K_{(\alpha)}^i(z) \) is a basis of holomorphic Killing vectors for the metrics \( g_{ij}^y(z, \overline{z}) \), this means:

\[
\partial_j K_{(\alpha)}^i(z) = 0 \Rightarrow \partial_j K_{(\alpha)}^i(\overline{z}) = 0 ; \quad K_{(\alpha)}^i = (K_{(\alpha)}^i)^* . \quad (222)
\]

Vectors \( K_{(\alpha)}^i \) are the generators of infinitesimal holomorphic coordinates transformations:

\[
\delta z^i = \varepsilon^K K_{(\alpha)}^i(z) \quad (223)
\]

that maintain invariant the metrics \( g_{ij}^y(z, \overline{z}) \).

The vector fields:

\[
K_{(\alpha)}^i = K_{(\alpha)}^i \partial_i \quad (224)
\]

associated to such Killing vectors close a Lie algebra:

\[
\left[ K_{(\alpha)}^i , K_{(\beta)}^j \right] = h_{\alpha\beta}^\gamma K_{(\gamma)}^i \quad (225)
\]

and the vectors can be normalized so that the structure constants are completely antisymmetric:

\[
h_{\alpha\beta}^\gamma = h_{\alpha\gamma}^\beta = h_{\beta\gamma}^\alpha . \quad (226)
\]

As the metrics \( g_{ij}^y(z, \overline{z}) \) is the derivative of other fundamental objects, so the Killing vectors of a Kähler manifold are the derivatives of a convenient prepotential:

\[
\overline{\partial} \overline{\partial} \varphi \quad (227)
\]
Therefore it is possible to define a Killing vector through a real function $\mathcal{P}_{(\alpha)}$ such that $i g^{\alpha \beta} \partial_{\beta} \mathcal{P}_{(\alpha)}$ is holomorphic. The infinitesimal transformation of holomorphic coordinates extended to fermions is an invariance of $\mathcal{L}_1$ because, under an isometry, the Kähler potential is not invariant, but changes by a Kähler transformation:

$$G(z + \delta z, \bar{z} + \delta \bar{z}) = G(z, \bar{z}) + \epsilon^{\alpha} \text{Re} f_{\alpha}(z), \quad (228)$$

that can be compensated in $\mathcal{L}_1$ by another Kähler transformation. The form of the isometry transformation on fermions is therefore:

$$\delta \chi^i = \epsilon^{\alpha} \partial_j K^i_{(\alpha)}(z) \chi^j - \frac{i}{2} \epsilon^{\alpha} f_{\alpha}(z) \chi^j; \quad (229)$$

$$\delta \psi_* = \frac{i}{2} \epsilon^{\alpha} \text{Im} f_{\alpha}(z) \psi_* . \quad (230)$$

Also the part $\Delta \mathcal{L}$ of the Lagrangian is not invariant under the isometry $K^i_{(\alpha)}(z)$, unless the compensating Kähler transformation is null:

$$f_{\alpha}(z) = 0 . \quad (231)$$

So we consider holomorphic vectors $K^i_{(\alpha)}(z)$ that satisfy the most restrictive condition of keeping invariant the Kähler potential:

$$\partial_j G K^i_{(\alpha)} + \partial_i G K^i_{(\alpha)} = 0 . \quad (232)$$

In particle physics applications we can have the situation in which the Killing vector is a linear function in $z$:

$$K^i_{(\alpha)} = (T_a)^i_j z^j \Rightarrow \delta z^j = \epsilon^{\alpha} (T_a)^i_j z^j . \quad (233)$$

In this case Eq. (229) becomes:

$$\delta \chi^i = \epsilon^{\alpha} (T_a)^i_j \chi^j . \quad (234)$$

In the case of linear isometries, the prepotential of Killing vectors is expressed in terms of the first derivative of the Kähler potential:

$$\mathcal{P}_{(\alpha)} = -i \partial_i G (T_a)^i_j \chi^j . \quad (235)$$

### 5.2. The vector multiplet

If we assume the existence of an $m$-dimensional isometry group $\mathfrak{S}$, is possible to introduce $m$ vector multiplets:
that belong to the adjoint representation of $\mathfrak{g}$.

$A^{\alpha}$ is a bosonic 1-form, $\lambda^{\alpha}$ is a Majorana spinor, whose chiral projections are given by:

$$\lambda^{\alpha} = \frac{1+\gamma_5}{2} \lambda^{\alpha}, \quad \lambda^{\alpha \ast} = \frac{1-\gamma_5}{2} \lambda^{\alpha};$$

$$\bar{\lambda}^{\alpha \ast} = (\lambda^{\alpha})^{\ast} \gamma_0; \quad \bar{\lambda}^{\alpha} = (\lambda^{\alpha \ast})^{\ast} \gamma_0. \quad (237)$$

We denote with $\hat{\nabla}$ the covariant derivative with respect to Lorentz transformations, the isometry group $\mathfrak{g}$, Kähler transformations and holomorphic diffeomorfisms. We fix:

$$\hat{\nabla} z^{\alpha} = dz^{\alpha} + A^{\alpha} (T^{\alpha \beta}) z^{\beta}, \quad (238)$$

$$\hat{\nabla} \lambda^{\alpha} = \nabla \lambda^{\alpha} + A^{\alpha} (T_{a}^{\alpha}) z^{\alpha}, \quad (239)$$

$$\hat{\nabla} \bar{\lambda}^{\alpha} = d\bar{\lambda}^{\alpha} - \frac{1}{4} \omega^{ab} \gamma_{ab} \bar{\lambda}^{\alpha \ast} + h^{\alpha \beta} A^{\beta} \bar{\lambda}^{\alpha \ast} + i \frac{1}{2} Q \bar{\lambda}^{\alpha \ast}, \quad (240)$$

$$\hat{\nabla} \psi^{\alpha} = \nabla \psi^{\alpha}, \quad (241)$$

having assumed that the Kähler weight of $\bar{\lambda}^{\alpha \ast}$ is equal to that of gravitino; this is consistent with Bianchi identities.

In the following we will continue to write $\nabla$ instead of $\hat{\nabla}$ for simplicity of notation; this will not create confusion. The idea is to replace the new covariant derivative $\hat{\nabla}$ everywhere in Bianchi identities.

If we want to resolve Bianchi identities even in the presence of vector supermultiplets, it is necessary to write the rheonomic parametrization of the curvatures associated with $A^{\alpha}$ and $\lambda^{\alpha}$. Indicating with $F^{\alpha}$ the curvature of $A^{\alpha}$:

$$F^{\alpha \def} = dA^{\alpha} + h^{\alpha \beta} A^{\beta} \wedge A^{\gamma}, \quad (242)$$

we write the following rheonomic parametrization:

$$F^{\alpha \ param} = F^{\alpha \ ab} V^{\alpha} \wedge V^{b} + \frac{i}{2} \bar{\lambda}^{\alpha \ast} \gamma_{m} \psi^{\ast} \wedge V^{m} + \frac{i}{2} \bar{\lambda}^{\alpha \ast} \gamma_{m} \psi \wedge V^{m}, \quad (243)$$

where the coefficient of $\psi \wedge V$ defines the supersimmetric partner $\bar{\lambda}^{\alpha \ast}$ of $A^{\alpha}$ and has been arbitrarily normalized. We write the rheonomic parametrization of $\bar{\lambda}^{\alpha \ast}$ in this way:

$$\nabla \bar{\lambda}^{\alpha \ast} = \nabla_{\alpha} \bar{\lambda}^{\alpha \ast} V^{\alpha} + \alpha F^{(+) \ ab} \gamma^{ab} \psi^{\ast} + i D^{\alpha} \psi^{\ast}, \quad (244)$$

$$\nabla \lambda^{\alpha \ast} = \nabla_{\alpha} \lambda^{\alpha \ast} V^{\alpha} + \alpha F^{(-) \ ab} \gamma^{ab} \psi^{\ast} - i D^{\alpha} \psi^{\ast}, \quad (245)$$
where $F^{(+)}_{ab}$ and $F^{(-)}_{ab}$ are the self-dual and antiself-dual part of $F^a_{ab}$ respectively:

$$F^a_{ab} = F^{(+)}_{ab} + F^{(-)}_{ab}, \quad (246)$$

$$\epsilon_{abcd} F^{(\pm)}_{ac} = \pm 2i F^{(\pm)}_{ab}. \quad (247)$$

It is convenient to use the following notation: we call “$(m, n)$ sector” of an equation among forms that with $m$ vierbeins $V$ and $n$ gravitinos $\Psi$.

From the Bianchi identity:

$$(\nabla F^a)_{(1,2)} = 0, \quad (248)$$

we get:

$$2i F^{a}_{ab} \overline{\psi} \gamma^{a} \psi \wedge V^{b} + \frac{i}{2} \overline{\psi} \gamma^{m} (\alpha F^{(+) a}_{ab} \gamma^{ab} +$$

$$+ i D^{a}) \gamma^{a} \wedge V^{m} + \frac{i}{2} (\alpha F^{(-) a}_{ab} \gamma^{ab} - i D^{a}) \gamma^{a} \wedge V^{m} = 0, \quad (249)$$

From (249) we obtain:

$$(D^{a})^{\dagger} = D^{a}, \quad (250)$$

real function, said “auxiliary field”, and:

$$2i (F^{(+) a}_{ab} + F^{(-) a}_{ab}) \overline{\psi} \gamma^{a} \psi \wedge V^{b} + 2i \alpha F^{(+) a}_{ab} \overline{\psi} \gamma^{a} \wedge$$

$$\wedge \delta^{ma} \gamma^{b} \psi \wedge V^{m} + 2i F^{(-) a}_{ab} \overline{\psi} \gamma^{a} \wedge \delta^{ma} \gamma^{b} \psi \wedge V^{m} = 0, \quad (251)$$

having considered Eq. (A.7.1). Then we obtain: $\alpha = 1$.

About curvatures of $z^i$ and $\chi^j$, we replace $d$ with $\nabla$. The definitions of “gauged” curvatures of $z^i$ and $\chi^j$ become:

$$R(z^i) = dz^i + A^a(T_{a})^{j}_{i} z^{param} = \overline{z}^{param} V^{a} + \overline{\psi} \gamma^{a} \psi; \quad (252)$$

$$\nabla \chi^{j} = \Phi \chi^{j} - \frac{i}{2} Q \chi^{j} + \Gamma^{j}_{ik} dz^{k} \chi^{i} + A^{a}(T_{a})^{j}_{i} \chi^{i}^{param}$$

$$= (\nabla \chi^{j})^{param} V^{a} + i Z^{a} \gamma^{a} \psi + \mathcal{K} i \psi. \quad (253)$$

The Bianchi identity of $z^i$ has the same previously found solution.
5.3. “Off-shell” parametrization of gravitino

Now we must resolve still the Bianchi identities of supergravity in the presence of both Wess-Zumino multiplet and vector multiplets.

We replace the new covariant derivative “∇” instead of “d”. Then the parametrization of gravitino curvature is:

\[ \rho = \rho_{ab} V^a \wedge V^b + i A_a \psi^* \wedge V^a + i A'_a \gamma^a \psi \wedge V^a + S \gamma \psi^* \wedge V^a. \] (254)

The corresponding Bianchi identity is given by:

\[ \mathcal{D} \rho + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi^* + \frac{1}{2} g_{ij} \nabla z^i \wedge \nabla z^j \wedge \psi^* = 0, \] (255)

with the “off-shell” parametrization of \( R^{ab} \):

\[ R^{ab} = R^{ab}_{cd} V^c \wedge V^d + \overline{\theta}^{ab} \psi^* \wedge V^c + \overline{\theta}^{ab} \gamma \psi \wedge V^c - \]

\[ -i S^* \psi \wedge \gamma^{ab} \psi^* + i S \overline{\psi}^* \wedge \gamma^{ab} \psi^* - 2i A'_a \overline{\psi}^* \wedge \gamma^d \psi^* \varepsilon_{abcd}. \] (256)

Considering that the parametrization of \( \nabla z^i \) is the same to that in absence of vector multiplets, the explicit form of gravitino Bianchi identity in the sector \( \psi \wedge \psi \wedge \psi \) has the previously found form, but is different because it is now possible to have also the current:

\[ Q_{ab} \overline{\lambda}^a \gamma^b \lambda^b. \] (257)

In the case of Wess-Zumino multiplets, i.e. without vector multiplets, one only current \( \overline{\lambda}^i \gamma_a \chi^j \) exists that can be identified with fields \( A_a \) and \( A'_a \). It is therefore not more possible to fix \( A_a = 0 \), but we write:

\[ A_a = Q_{ab} \overline{\lambda}^a \gamma^b \lambda^b \] (258)

and therefore:

\[ A'_a = \frac{1}{2} Q_{ab} \overline{\lambda}^a \gamma^b \lambda^b + \frac{1}{8} g_{ij} \overline{\lambda}^i \gamma^j \chi^j. \] (259)

Let us consider now the (1,2) sector of the same Eq. (255):

\[ 2i \rho_{ab} \overline{\psi}^* \wedge \gamma^a \psi \wedge V^b - i \psi \wedge (\overline{\psi}, \nabla_{(0,1)} A_a + \overline{\psi}^* \nabla^*_{(0,1)} A_a) \wedge V^b - \]

\[ -i \gamma^a_b \psi^* \wedge (\overline{\psi}, \nabla_{(0,1)} A'_a + \overline{\psi}^* \nabla^*_{(0,1)} A'_a) \wedge V^b - \gamma^a_b \psi^* \wedge \]

\[ \wedge (\nabla_m S \overline{\psi}^* \chi^m + \nabla_m S \overline{\psi}^* \chi^m) \wedge V^b + \frac{1}{4} \gamma_{ab} \psi \wedge (\overline{\psi}, \theta^a_{bc} + \]

\[ \left( \overline{\lambda}^a \gamma^b \lambda^b \right) \wedge V^b. \]
The cancellation of the current $\bar{\psi}^* \wedge \gamma^m \psi^*$ brings, as previously, to: $S = ie e^{G/2}$.

In $\bar{\psi} \wedge \gamma^m \psi^*$ sector we have:

$$i \frac{\bar{\psi} \wedge \gamma^m \psi^*}{8} (\nabla_{(0,1)\cdot} \mathbf{A}_b) + i \frac{\bar{\psi} \wedge \gamma^m \psi^*}{8} (\nabla_{(0,1)\cdot} \mathbf{A}'_a) \bar{\psi} \wedge \gamma^m \psi^* = 0.$$ (261)

Multiplying both members with $\gamma^m$ and considering Eqs (A.7.4), (A.7.6) we get:

$$-3 \frac{\bar{\psi} \wedge \gamma^m \psi^*}{2} (\nabla_{(0,1)\cdot} \mathbf{A}_b) + i \frac{\bar{\psi} \wedge \gamma^m \psi^*}{2} (\nabla_{(0,1)\cdot} \mathbf{A}'_a) - \frac{1}{16} g_{ij} \bar{\psi} \gamma^i \gamma^j b \bar{\psi}_a \wedge \gamma^m \psi^* = 0,$$ (262)

with $\mathbf{A}_a$, $\mathbf{A}'_a$ and $\Theta^{ac}_b$ given by (258), (259) respectively,

$$\Theta^{ac}_b = -2 i \gamma^i (a \rho c b) + i \gamma^b \rho^{ac}_b,$$ (263)

and $Q_{\alpha \beta}$ in general function of the fields $z^i$, $\bar{z}^{i'}$.

In the sector of one-index current of Eq. (255) we have:

$$2 i \rho_{ab} - \frac{1}{2} \gamma^i (\nabla^i_{(0,1)\cdot} \mathbf{A}_a) - \frac{1}{2} \gamma^i (\nabla^i_{(0,1)\cdot} \mathbf{A}'_a) - \frac{ie}{4} \gamma^i \gamma^m (\partial_m G) e^{G/2} +$$

$$+ \frac{1}{8} \gamma^i \gamma_a \Theta^{* i m}_b - \frac{1}{4} b g_{ij} \bar{z}^i \gamma_a \wedge \gamma^j = 0.$$ (264)

Multiplying Eq. (264) with $\gamma^a$, we get the motion equation of gravitino, that now includes the coupling to vector multiplets too.

It results indeed:

$$2 i \gamma^a \rho_{ab} - 2 i (\nabla^a_{(0,1)\cdot} \mathbf{A}_b) + \frac{ie}{2} \gamma^m (\partial_m G) e^{G/2} - g_{ij} \bar{z}^i \gamma_a \wedge \gamma^j = 0,$$ (265)

having used Eqs (A.2.4), (A.7.5), (A.2.5).

The spinorial derivative of $\mathbf{A}_b$ results:

$$(\nabla^i_{(0,1)\cdot} \mathbf{A}_b) = \nabla^i_{(0,1)\cdot} (Q_{a\beta} \bar{\lambda}^{a\beta}_b) = (\partial_i Q_{a\beta} \chi^i + \partial_{a\beta} Q_{a\beta} \chi^i) \bar{\lambda}^{a\beta}_b +$$

$$+ Q_{a\beta} (-4 F_{\beta \alpha}^i a \gamma^a + i D^{\beta}_{a\beta} \gamma_a) \bar{\lambda}^{a\beta}_b,$$ (266)
considering the parametrization of $\lambda^{a\cdot}$, $\lambda^{b\cdot}$ and that it holds:

$$(\nabla_{(0,1)}), \lambda^{a\cdot} = 0, \quad (267)$$

$$(\nabla_{(0,1)}^\ast) \lambda^{a\cdot} = 0. \quad (268)$$

The motion equation results:

$$2i \gamma^a \rho_{ab} - 2i (\partial_i Q_{ab} \chi^i + \partial_i Q_{ab}^\ast \lambda^i) \overline{\lambda} a \cdot \gamma^b \chi^{b\cdot} +$$

$$+ Q_{ab} (\overline{\lambda} a \cdot \gamma^b \gamma^c + i D^b \gamma_c) \chi^{a\cdot} + \frac{i e}{2} \gamma_b \chi^a (\partial \ G) \ e^{g/2} -$$

$$- g_{ab} Z^j_{a\cdot} \chi^j = 0. \quad (269)$$

Eq. (269) is still not explicit, because we don't know the form of fields $Q_{ab}$ and $D^b$, that will be determined in the following.

5.4. Bianchi identities of $z^i$ and $\chi^i$

We now analyze the new Bianchi identities of the Wess-Zumino multiplet. About $\chi^i$ we have:

$$\nabla \chi^i = \nabla \chi^i = \gamma^{i\cdot} a \cdot \nabla a \cdot \chi^i + A^a (T_a)_{/i} \chi^i =$$

$$= (\nabla \chi^i)_a V^a + i P^i a \gamma^a \psi^* + \nabla \psi^* + i L^i_{/ab} \gamma_{ab} \psi^*. \quad (270)$$

The Bianchi identity of $R(z^i)$ results:

$$\nabla^2 z^i = F^a (T_a)_{/i} z^j = \overline{\chi} a \cdot \chi^i + V^a + i \gamma^a \psi^* = \overline{\psi} \cdot \chi^i = \nabla \chi^i + \overline{\chi} a \cdot \psi^*. \quad (271)$$

The (0,2) sector brings (as previously) to:

$$P^i a = Z^i a; \quad \psi^* = \text{free}; \quad L^i_{/ab} = 0. \quad (272)$$

By the (0,1) sector it is possible to get the spinorial derivative of $Z^i a$:

$$(\nabla_{(0,1)} Z^i a) = S_{/i} a \cdot \chi^i - \frac{i}{2} \gamma^a \lambda^{a\cdot} (T_a)_{/i} z^j; \quad (273)$$

$$(\nabla_{(0,1)} Z^i a) = \nabla a \cdot \chi^i - i A^a a \cdot \chi^i + i A^b a \cdot \chi^i - \frac{i}{2} \gamma^a \lambda^{a\cdot} (T_a)_{/i} z^j. \quad (274)$$
About $\nabla_{(0,1)} \bar{Z}^{i'}_{\alpha}$ we get similarly:

$$(\nabla_{(0,1)} \bar{Z}^{i'}_{\alpha})_* = -S^* \gamma_a \chi^{i'} - \frac{i}{2} \gamma_a \lambda^{a*} (T_\alpha)^{i'}_{j'} z^{j'}; \quad (275)$$

$$(\nabla_{(0,1)} \bar{Z}^{i'}_{\alpha}) = \nabla_a \chi^{i'} - i A_a \chi^{i'} + i A^b_a \gamma^a \chi^{i'} - \frac{i}{2} \gamma_a \lambda^{a*} (T_\alpha)^{i'}_{j'} z^{j'}. \quad (276)$$

Let us consider now the Bianchi identity of $\chi^i$:

$$\nabla^2 \chi^i = \nabla(\nabla_a \chi^i V^a + i Z^a \gamma^a \psi^* + \mathfrak{K}^i \psi^*). \quad (277)$$

We note that the auxiliary field $\mathfrak{K}^i$ appearing in parametrization (277) does not coincide in general with the previously found value (Eq. (203)) because of the presence of vector multiplets. We write therefore:

$$\mathfrak{K}^i = \mathfrak{K}^i + \Delta \mathfrak{K}^i, \quad (278)$$

where $\mathfrak{K}^i$ is the expression (203) in absence of gauge fields, whereas $\Delta \mathfrak{K}^i$ has in general the form:

$$\Delta \mathfrak{K}^i = N^i_{\alpha \beta}(z, \bar{z}) \overline{\lambda}^{\alpha*} \lambda^{\beta*} + M^i_{\alpha \beta}(z, \bar{z}) \overline{\lambda}^{\alpha*} \lambda^{\beta*}. \quad (279)$$

that results to be the rheonomic form compatible with the rigid scale “-1” and with the Kähler scale “-1”.

Calculating the (0,2) sector of (277) and considering the $\overline{\psi}^* \gamma_{ab} \psi^*$ sector, we obtain:

$$\frac{1}{8} \nabla_l \mathfrak{K}^i - \frac{i}{4} \delta^i_{\beta} S^* = 0. \quad (280)$$

This equation is identically satisfied by the values of $\mathfrak{K}^i$ and $S$ previously found (Eqs (202), (203)). Furthermore we have:

$$(\nabla_{(0,1)} \Delta \mathfrak{K}^i)_* = 0; \quad (281)$$

$$\Rightarrow \nabla_{(0,1)} \times (N^i_{\alpha \beta}(z, \bar{z}) \overline{\lambda}^{\alpha*} \lambda^{\beta*} + M^i_{\alpha \beta}(z, \bar{z}) \overline{\lambda}^{\alpha*} \lambda^{\beta*}) = 0. \quad (282)$$

By Eq. (282), considering that (267) holds, it is:

$$M^i_{\alpha \beta} = 0; \quad (283)$$

$$\nabla_l N^i_{\alpha \beta} = 0. \quad (284)$$
Multiplying both members of (284) with $g_{\theta \theta}$ we get: 
\[ \nabla_i \left( g_{\theta \theta} N'_{\alpha \beta} \right) = 0 = \nabla_t N'_{\alpha \beta} = \partial_t N_{\alpha \beta} + \text{in which the Riemann affine connection appears, with indices that are never all starry or not starry. This implies that all these additional terms are null; therefore it results:} \]
\[ \partial_t N_{\alpha \beta} = 0, \quad (285) \]
i.e. $N_{\alpha \beta}$ is an anti-holomorphic function.

In the $\bar{\psi}^* \wedge \gamma^{ab} \psi^*$ sector, considering the validity of (A.7.5), we have identically zero; the same happens for the term: $\gamma^a \psi^* \wedge \bar{\psi}^* \gamma_a \lambda^a \cdot (T_a)^i_j z^j$.

Also the term $\gamma_{ab} \mathcal{K}^i \bar{\psi}^* \wedge \gamma^{ab} \psi^*$ is zero because $\gamma_{ab} \mathcal{K}^i$ and $\bar{\psi}^* \wedge \gamma^{ab} \psi^*$ have opposite self-duality.

The study of $\bar{\psi}^* \wedge \gamma^{i} \psi$ sector, using (A.2.4), (A.2.5), (A.7.1), (A.7.3), brings to an implicit motion equation, where functions $N'_{\alpha \beta}$ and $D^\beta$ are not known.

5.5. Bianchi identity of $F^\alpha$

We now analyze the Bianchi identities of vector multiplet of relevant interest. Considering the Fierz identities, the (0,3) sector gives $0 = 0$ identically and the same is for the (1,2) sector.

The (2,1) sector gives the spinorial derivative of $F^\alpha_{ab}$. Using (A.1.8), (A.7.1), (247) and contracting $(\nabla_{(0,1)} F^\alpha_{ab})^{(+)}$ with $\gamma^{ab}$, we get:

\[ (\nabla_{(0,1)} F^\alpha_{ab} \gamma^{ab})^{(\pm)} = \frac{i}{2} \gamma^{ab} \gamma_a \nabla_b \lambda^{a*} - 6 i S^e \lambda^{a*} - \frac{3}{2} \Lambda^a \gamma_a \lambda^{a*} - \frac{3}{2} \Lambda^a \gamma_a \lambda^{a*}. \quad (286) \]

We note also that:

\[ (\nabla_{(0,1)} F^\alpha_{ab})^{(+)} = (\nabla_{(0,1)} F^\alpha_{ab})^{(+)} = (287) \]

\[ (\nabla_{(0,1)} F^\alpha_{ab})^{(\pm)} = (\nabla_{(0,1)} F^\alpha_{ab})^{(\pm)} = (288) \]

because:

\[ (\nabla_{(0,1)} F^\alpha_{ab})^{(-)} = 0, \quad (289) \]

\[ (\nabla_{(0,1)} F^\alpha_{ab})^{(+)} = 0. \quad (290) \]
5.6. Bianchi identity of gaugino

The Bianchi identity of gaugino is given by:

\[ \nabla^2 \lambda^a = \nabla(\nabla_a \lambda^a \nabla^a + F^{ab} \gamma^b \psi_\alpha + i D^a \psi_\alpha) \quad \text{(291)} \]

Considering the (0,2) sector of (291) and of this the \( \overline{\psi} \wedge \gamma^a \psi_\alpha \) sector, we get:

\[ \frac{i}{4} \gamma^{lm} \lambda^a \nabla^a \overline{\psi} \wedge \gamma^{lm} \psi_\alpha = \frac{1}{8} \gamma^{ab} \gamma^{lm} \left( \nabla^{(0,1)} \cdot F^{ab} \right) \overline{\psi} \wedge \gamma^{lm} \psi_\alpha + \]

\[ + \frac{i}{8} \gamma^{lm} \left( \nabla^{(0,1)} \cdot D^a \right) \overline{\psi} \wedge \gamma^{lm} \psi_\alpha \quad \text{(292)} \]

Multiplying both members of (292) with \( \gamma^{lm} \) and using (A.7.4) and (A.7.6), we get:

\[ -3i \lambda^a \nabla^a = \frac{1}{2} \gamma^{ab} \left( \nabla^{(0,1)} \cdot F^{ab} \right) + \frac{3}{2} i \left( \nabla^{(0,1)} \cdot D^a \right) \quad \text{(293)} \]

We make now the following ansatz, which is the only compatible with rheonomy, Kähler weights “0” and “- 1”:

\[ D^a = D^a \lambda^{\alpha} + L^{\alpha} \beta \overline{\lambda} \beta + L^{\alpha} \beta \overline{\lambda} \beta \quad \text{(294)} \]

Calculating the spinorial derivative of \( \nabla \lambda^a \), i.e. \( (\nabla \lambda^a) \), in two different ways:

a) using the motion equation of \( \lambda^a \);

b) directly by the Bianchi identity of gaugino (the complex conjugate of Eq. (291)) in the (1,1) sector, and matching the two results, informations on introduced arbitrary functions \( L^{\alpha} \beta \), \( Q_{\alpha \beta} \) and \( N^i \alpha_\beta \) are obtained. Remembering Eqs (202), (203) and writing:

\[ D^a = (M^{-1})^{\alpha } \beta \quad \text{(295)} \]

with \( \beta \) prepotenzial of Killing vectors (227), in the case of \( m \) multiplets, we get:

\[ L^{\alpha} \beta = -\frac{i}{2} (M^{-1})^{\alpha } \beta \partial_i M^\prime_\beta \quad \text{(296)} \]
The $M$ function results to be a harmonic function, i.e. the real part of an analytic function:

$$M = \frac{1}{2}(f(z) + \bar{f}(\bar{z})) ; \quad (297)$$

then we have:

$$\partial^*_k M = \frac{1}{2} \partial^*_k \bar{f}(\bar{z}) , \quad (298)$$

and:

$$N^*_k = \frac{1}{4} \partial^*_k \bar{f}(\bar{z}) . \quad (299)$$

The found result is generalizable to the case of $m$ vector multiplets:

$$N^*_{k\alpha\beta} = \frac{1}{4} \partial^*_k \bar{f}_{\alpha\beta}(\bar{z}) . \quad (300)$$

With the determination of $Q_{\alpha\beta}$, all the unknown functions appearing in the initial ansatz on the parameterization of curvatures are determined in terms of the analytic function $f_{\alpha\beta}(z)$. To determine the latter one, we analyze a sector in which the auxiliary fields of supergravity explicitly appear. We get:

$$Q_{\alpha\beta} = M_{\alpha\beta} = \text{Re} f_{\alpha\beta} . \quad (301)$$
Chapter VI

CONCLUSIONS

Concluding, the obtained results by the analysis of the Bianchi identity of gaugino equation can be summarized as follows:

→ by equation:

\[
(\nabla_{(0,1)}(\nabla \Lambda^{a*}))_{\text{Equations of motion}} = (\nabla_{(0,1)}(\nabla \Lambda^{a*}))_{\text{Bianchi identity}}
\]

we get, in the \(1\psi\) sector:

→ Bosonic terms:

\[
L^{a}_{\beta \lambda} = -\frac{i}{2} (M^{-1})^{a}_{\rho \lambda} \partial_{i} M^{i \beta}.
\]

→ Bilinear terms in \(\chi \chi 1 \psi\): we get an equation that is identically satisfied with the previous positions.

→ Bilinear terms in \(\chi \lambda 1 \psi, \bar{Z}^{i}_{a}\):

\[
\partial_{i} \partial_{i} M^{a \beta} = 0.
\]

→ Terms in \(\lambda \lambda \lambda\) boson \(1 \psi\):

\[
N_{k\lambda \alpha \beta} = \frac{1}{4} \partial_{k} \bar{f}_{\alpha \beta}(\bar{z}),
\]

where:

\[
M_{a \beta} = \text{Re} f_{a \beta}.
\]

→ \(\chi \lambda \lambda \lambda\) sector:

\[
Q_{a \beta} = \text{Re} f_{a \beta}.
\]

The motion equations of \(\chi^{i}\) and \(\lambda^{a*}\), written in implicit form, can be written in explicit form, having defined the form of auxiliary fields \(\chi^{i}, D^{a}, S, A_{a}, A'_{a}\).

For deriving the motion equations of all fields of the theory, it is easier to use the variational principle, writing the complete Lagrangian that includes the interaction of the system “supergravity + Wess-Zumino multiplets” with “vector multiplets”. The determination of auxiliary fields makes relatively simple the construction of the complete Lagrangian.
The complete theory of $D = 4$, $N = 1$ supergravity, and therefore of the 4-dimensional heterotic superstring, ultimately contains two arbitrary functions:

a) the first is the real function $G(z, \bar{z})$, i.e. the Kähler potential of the Kähler manifold of Wess-Zumino multiplets;

b) the second is the analytic function $f_{\alpha \beta}(z)$.

Informations on them can be obtained by the analysis of the fundamental string theory, of which supergravity is the effective theory.

Found results, obtained in the context of a geometric formulation of the “supergravity + matter” coupling, are in complete agreement with the results obtained by means of the superconformal tensor calculus in the component approach [1,35-40].
REFERENCES


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APPENDIX

A.1. Gamma matrices

Gamma matrices satisfy to anti-commutation relations:

\[ \{ \gamma^a, \gamma^b \} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab} \]

\[ \eta^{ab} = \text{diag} (1, -1, -1, -1) \quad (a, b = 0, \ldots, 3) \quad (A.1.1) \]

and to Hermitian conditions:

\[ (\gamma^a)^+ = \gamma^0 \gamma^a \gamma^0 \quad (A.1.2) \]

with \( \gamma^0 \) Hermitian and \( \gamma^i \) anti-Hermitians. They are related to \( \beta \) and \( \alpha \) matrices in the following way:

\[ \gamma^0 = \beta; \quad \gamma^i = \beta \alpha^i \quad (A.1.3) \]

where \( \beta=(\alpha_1, \alpha_2, \alpha_3) \) and \( \beta \) is a 4 x 4 Hermitian matrix satisfying to:

\[ \{ \alpha_i, \alpha_j \} = 2 \delta_{ij}; \quad \{ \alpha_i, \beta \} = 0; \quad \beta^2 = 1; \quad i, j = 1, 2, 3. \quad (A.1.4) \]

The \( \gamma^5 \) matrix is defined as:

\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{i}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d \quad (A.1.5) \]

with the following properties:

\[ \{ \gamma^a, \gamma^5 \} = 0; \quad (\gamma^5)^2 = 1; \quad (\gamma^5)^+ = (\gamma^5) \quad (A.1.6) \]

\[ \gamma^0 \gamma^5 \gamma^0 = -\gamma_5^+. \quad (A.1.7) \]

\( \gamma_{ab} \) matrices, that are the anti-symmetric part of \( \gamma_a \gamma_b \):

\[ \gamma_a \gamma_b = \gamma_{ab} + \delta_{ab}, \quad (A.1.8) \]

\[ \gamma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b], \quad (A.1.9) \]

satisfy to:

\[ (\gamma^{ab})^+ = \gamma^0 \gamma^{ab} \gamma^0. \quad (A.1.10) \]
\( \varepsilon^{abcd} \) is the totally anti-symmetric tensor of Levi-Civita:

\[
\varepsilon^{abcd} = \begin{cases} +1 & (a) \\ -1 & (b) \\ 0 & (c) \end{cases}, \quad (A.1.11)
\]

(a): \( \{a, b, c, d\} \) are a even permutation of \( \{0, 1, 2, 3\} \),
(b): \( \{a, b, c, d\} \) are an odd permutation of \( \{0, 1, 2, 3\} \),
(c): two or more indices are equal.

### A.2. Contraction identities

About \( \varepsilon^{abcd} \), the following identities hold:

\[
\varepsilon^{abcd} \varepsilon_{abmn} = -2(\delta^e_m \delta^d_n - \delta^e_n \delta^d_m); \quad (A.2.1)
\]

\[
\varepsilon^{abcd} \varepsilon_{abc} = -6 \delta^d_m; \quad (A.2.2)
\]

\[
\varepsilon^{abcd} \varepsilon_{abcd} = -24. \quad (A.2.3)
\]

About \( \gamma \) matrices, it holds:

\[
\gamma_a \gamma^a = 4, \quad (A.2.4)
\]

\[
\gamma_a \gamma^b \gamma^a = -2 \gamma^b, \quad (A.2.5)
\]

\[
\gamma_a \gamma^b \gamma^c \gamma^a = 4 \delta^b_c, \quad (A.2.6)
\]

\[
\gamma_a \gamma^b \gamma^c \gamma^d \gamma^a = -2 \gamma^d \gamma^b, \quad (A.2.7)
\]

\[
\gamma_a \gamma^b \gamma^c \gamma^d \gamma^f \gamma^a = 2(\gamma^f \gamma^b \gamma^c \gamma^d + \gamma^d \gamma^c \gamma^b \gamma^f). \quad (A.2.8)
\]

### A.3. Traces

(a) For each pair of \( n \times n \) matrices “A” and “B”, it holds:

\[
Tr(AB) = Tr(BA). \quad (A.3.1)
\]

(b) If \( (\gamma^a \gamma^b \ldots \gamma^m \gamma^n) \) contains an odd number of \( \gamma \) matrices, it holds:

\[
Tr(\gamma^a \gamma^b \ldots \gamma^m \gamma^n) = 0, \text{ in particular: } Tr(\gamma^a) = 0. \quad (A.3.2)
\]

(c) For a product of an even number of \( \gamma \) matrices, we have:
\[ \text{Tr}(\gamma^a \gamma^b) = 4 \delta^{ab}, \quad (A.3.3) \]
\[ \text{Tr}(\gamma^{ab}) = 0, \quad (A.3.4) \]
\[ \text{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = 4(\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}). \quad (A.3.5) \]

(d) For every product of \( \gamma \) matrices, it is:
\[ \text{Tr}(\gamma^a \gamma^b \ldots \gamma^m \gamma^n) = \text{Tr}(\gamma^n \gamma^m \ldots \gamma^b \gamma^a). \quad (A.3.6) \]

(e) In products involving \( \gamma^5 \) matrix, the most important relations are:
\[ \text{Tr}(\gamma^5) = 0, \quad (A.3.7) \]
\[ \text{Tr}(\gamma^5 \gamma^a) = \text{Tr}(\gamma^5 \gamma^a \gamma^b) = \text{Tr}(\gamma^5 \gamma^a \gamma^b \gamma^c) = 0, \quad (A.3.8) \]
\[ \text{Tr}(\gamma^5 \gamma^a \gamma^b \gamma^c \gamma^d) = -4i \epsilon^{abcd}. \quad (A.3.9) \]

A.4. Charge conjugation matrix

It is the \( C \) matrix with the following features:
\[ C^T = -C, \quad (A.4.1) \]
\[ C^2 = -1. \quad (A.4.2) \]

The following relations hold:
\[ C \gamma_a C^{-1} = -\gamma_a^T, \quad (A.4.3) \]
\[ C \gamma_5 C^{-1} = \gamma_5^T, \quad (A.4.4) \]
\[ C \gamma_{ab} C^{-1} = -\gamma_{ab}^T, \quad (A.4.5) \]
\[ C(\gamma_5 \gamma_a) C^{-1} = (\gamma_5 \gamma_a)^T. \quad (A.4.6) \]

A.5. Bilinear covariants

The five fundamental bilinear covariants of Dirac theory are:
\[ \overline{\psi} \psi; \quad \overline{\psi} \gamma^a \psi; \quad \overline{\psi} \gamma^{ab} \psi; \quad \overline{\psi} \gamma^5 \psi; \quad \overline{\psi} \gamma^a \gamma^5 \psi. \quad (A.5.1) \]

Under a Lorentz transformation, they transform as:
\[ \overline{\psi} \rightarrow \text{a scalar} \quad (A.5.2a) \]

\[ \overline{\psi} \gamma^a \psi \rightarrow \text{a vector} \quad (A.5.2b) \]

\[ \overline{\psi} \gamma^{ab} \psi \rightarrow \text{an anti-symmetric tensor of rank 2} \quad (A.5.2c) \]

\[ \overline{\psi} \gamma^5 \psi \rightarrow \text{a pseudo-scalar} \quad (A.5.2d) \]

\[ \overline{\psi} \gamma^5 \gamma^a \psi \rightarrow \text{a pseudo-vector} \quad (A.5.2e) \]

In spinorial indices, matrices:

\[ C, C \gamma_s, C \gamma_s \gamma_a \text{ are anti-symmetric,} \quad (A.5.3) \]

\[ C \gamma_a, C \gamma_{ab} \text{ are symmetric.} \quad (A.5.4) \]

It follows that:

\[ \overline{\psi} \gamma_s \gamma_a \psi = \overline{\psi} \gamma_s \psi = \overline{\psi} \psi = 0. \quad (A.5.5) \]

A.6. Particular representations of gamma matrices

If \( \gamma^a (a = 0, \ldots, 3) \) and \( \gamma^5 (a = 0, \ldots, 3) \) are two series of matrices that satisfy relations (A.1.1) and (A.1.2), the fundamental Pauli theorem states:

\[ \gamma^a = U \gamma^a U^+, \text{ where } U \text{ is a unitary matrix} \quad (A.6.1) \]

Two particular useful representations are:

(a) **Pauli-Dirac representation**: Dirac matrices can be written as:

\[
\begin{align*}
\gamma^0 &= \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\alpha_k &= \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad (A.6.2) \\
\gamma^k &= \beta \alpha^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad (A.6.3) 
\end{align*}
\]

from which we get:

\[
\sigma^{ij} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad i, j, k = 1, 2, 3 \text{ in cyclic order} \quad (A.6.4)
\]
\[ \sigma^{0i} = i\alpha^k = i\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad (A.6.5) \]

and:

\[ \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (A.6.6) \]

(b) Majorana representation: \( \gamma^a_M = -\gamma^a M^* \) matrices \((a = 0, \ldots, 3)\) are obtained by Pauli-Dirac representation through:

\[ \gamma^a M = U \gamma^a U^+ \quad (A.6.7) \]

with:

\[ U = U^{-1} = U^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & (1 + \gamma^2) \end{pmatrix}. \quad (A.6.8) \]

Explicitly we have:

\[ \gamma^0 M = \gamma^0 \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad (A.6.9a) \]

\[ \gamma^1 M = \gamma^2 \gamma^1 = i \sigma^{12} = \begin{pmatrix} i \sigma_3 & 0 \\ 0 & i \sigma_3 \end{pmatrix}, \quad (A.6.9b) \]

\[ \gamma^2 M = -\gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad (A.6.10a) \]

\[ \gamma^3 M = \gamma^2 \gamma^3 = -i \sigma^{23} = \begin{pmatrix} -i \sigma_1 & 0 \\ 0 & -i \sigma_1 \end{pmatrix}, \quad (A.6.10b) \]

\[ \gamma^5 M = -i \gamma^0 \gamma^1 \gamma^3 \gamma^5 M = \gamma^0 \sigma^{31} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \quad (A.6.11) \]

In this representation all matrices are pure imaginary, because Pauli matrices \( \sigma^1, \sigma^3 \) are real and \( \sigma^2 \) is pure imaginary.

A.7. Useful formulas for the expansion of the algebra of gamma matrices

We report some useful relations in the expansion of calculations involving the algebra of gamma matrices:
\( \gamma^a \gamma^c = 2 \delta^a_{[b} \gamma^c_{c]} + \gamma^a_{bc} = 2 \delta^a_{[b} \gamma^c_{c]} + i \epsilon^{a b c d} \gamma^d, \quad (A.7.1) \)

\( \gamma^a_{bc} = -2 \delta^{[a}_{b} \gamma^c_{c]} + \gamma^a_{bc} = -2 \delta^{[a}_{b} \gamma^c_{c]} + i \epsilon^{a b c d} \gamma^d, \quad (A.7.2) \)

\( \gamma^5 \gamma^a = -\frac{i}{2} \epsilon_{abcd} \gamma^{cd} = \gamma^a \gamma^5, \quad (A.7.3) \)

\( \gamma^a_{\gamma^a} = i \epsilon^{a b c d} \gamma^b_{\gamma^a} = -4 \delta^a_{\Delta^b [a} \gamma^b_{c]} - 2 \delta^a_{\gamma^a}, \quad (A.7.4) \)

\( \gamma^a_{\gamma^a} \gamma^a_{m} = \gamma^a_{\gamma^a} \gamma^m_{a} \gamma^a_{ab} = 0, \quad (A.7.5) \)

\( \gamma^a_{\gamma^a} \gamma^a_{cd} \gamma^a_{ab} = 4 \gamma^a_{cd}, \quad (A.7.6) \)

\( \epsilon_{a b c} \gamma^{a b c} = 3 i \gamma^5 \gamma^f, \quad (A.7.7) \)

\( \epsilon_{a b c f} \gamma^{a b c f} = -2 i \gamma^5 \gamma^a_{\gamma^b b}, \quad (A.7.8) \)

\( \psi \gamma^a \psi \overline{\psi} \gamma^b \psi = \overline{\psi} \gamma^5 \gamma^a \psi \overline{\psi} \gamma^b \psi = 0, \quad (A.7.9) \)

\( \psi \overline{\psi} \gamma^a \psi = -2 i \Sigma_a^{(12)}, \quad (A.7.10) \)

\( \psi \overline{\psi} \gamma^a \psi = -2 i \Sigma_a^{(8)} + 2 i \gamma^a \Sigma_{ab}^{(12)}, \quad (A.7.11) \)

Re-defining: \(-2 i \Xi \rightarrow \Xi\), we have:

\( \psi \overline{\psi} \gamma^a \psi = \Xi_a^{(12)}, \quad (A.7.12) \)

\( \psi \overline{\psi} \gamma^a \psi = \Xi_a^{(8)} - \gamma^a \Xi_b^{(12)}, \quad (A.7.13) \)

with:

\( \gamma^a \Xi_a^{(12)} = 0, \quad (A.7.14) \)

\( \gamma^b \Xi_{ab}^{(8)} = 0. \quad (A.7.15) \)

It follows:

\( \gamma^a \psi \overline{\psi} \gamma^a \psi = \Xi_b^{(12)}, \quad (A.7.16) \)

\( \gamma^a \psi \overline{\psi} \gamma^a \psi = \gamma^a \Xi_b^{(12)} = -\Xi_b^{(12)a}. \quad (A.7.17) \)
A.8. Properties of fermionic currents

If \( \chi_A \) is a 0-form Majorana spinor with spin 1/2 and \( \xi_A \) a 1-form Majorana spinor with spin 3/2, we have:

\[
\lambda_d = \frac{1 + \gamma_5}{2} \chi_d; \quad \psi_A = \frac{1 + \gamma_5}{2} \xi_A, \quad (A.8.1-2)
\]

\[
\lambda^t = \frac{1 - \gamma_5}{2} \chi_d; \quad \psi^t = \frac{1 - \gamma_5}{2} \xi_A, \quad (A.8.3-4)
\]

\[
\overline{\chi}_d = \overline{\chi}_d \frac{1 + \gamma_5}{2}; \quad \overline{\psi}_A = \overline{\chi}_d \frac{1 + \gamma_5}{2}, \quad (A.8.5-6)
\]

\[
\overline{\chi}^t = \overline{\chi}_d \frac{1 - \gamma_5}{2}; \quad \overline{\psi}^t = \overline{\chi}_d \frac{1 - \gamma_5}{2}, \quad (A.8.7-8)
\]

\[
(\overline{\chi}_d \lambda_B)^* = \overline{\chi}^B \lambda^d = \overline{\chi}^d \lambda^B, \quad (A.8.9)
\]

\[
(\overline{\chi}^t \gamma^a \lambda_B)^* = \overline{\chi}^B \gamma^a \lambda_d = -\overline{\chi}_d \gamma^a \lambda_B, \quad (A.8.10)
\]

\[
(\overline{\chi}^t \gamma^{ab} \lambda_B)^* = -\overline{\chi}_d \gamma^{ab} \lambda_d = \overline{\chi}^d \gamma^{ab} \lambda_B, \quad (A.8.11)
\]

\[
(\overline{\psi}_A \psi_B)^* = -\overline{\psi}^B \psi^A = \overline{\psi}^A \psi^B, \quad (A.8.12)
\]

\[
(\overline{\psi}^A \gamma^a \psi_B)^* = -\overline{\psi}_B \gamma^a \psi_A = -\overline{\psi}_A \gamma^a \psi^B, \quad (A.8.13)
\]

\[
(\overline{\psi}^A \gamma^{ab} \psi_B)^* = \overline{\psi}_B \gamma^{ab} \psi_A = \overline{\psi}_A \gamma^{ab} \psi^B. \quad (A.8.14)
\]

A.9. Representations of SO(1,3)

<table>
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<tr>
<th>Type of representation</th>
<th>Dimension</th>
<th>Corresponding tensor, spinor or spinor-tensor</th>
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<td>( X_{ab} = -X_{ba} ) (anti-symmetric tensor)</td>
</tr>
<tr>
<td>[1,0][+]</td>
<td>4</td>
<td>( X^a ) (vector)</td>
</tr>
<tr>
<td>[1,0][-]</td>
<td>4</td>
<td>( X^a ) (axial vector)</td>
</tr>
<tr>
<td>[0,0][+]</td>
<td>1</td>
<td>( X^+ ) (scalar)</td>
</tr>
<tr>
<td>[0,0][-]</td>
<td>1</td>
<td>( X^- ) (pseudo-scalar)</td>
</tr>
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<td>$[3/2,3/2]$</td>
<td>8</td>
<td>$\Xi_{ab} = -\Xi_{ba}$; $\gamma^b \Xi_{ab} = 0$ (irreducible spinor-tensor)</td>
</tr>
<tr>
<td>-------------</td>
<td>---</td>
<td>------------------------------------------------------------------</td>
</tr>
<tr>
<td>$[3/2,1/2]^{(+)}$</td>
<td>12</td>
<td>$\Xi_a^{(+)}$; $\gamma^a \Xi_a^{(+)} = 0$ (irreducible spinor-vector)</td>
</tr>
<tr>
<td>$[3/2,1/2]^{(-)}$</td>
<td>12</td>
<td>$\Xi_a^{(-)}$; $\gamma^a \Xi_a^{(-)} = 0$ (irreducible spinor-axial vector)</td>
</tr>
<tr>
<td>$[1/2,1/2]^{(+)}$</td>
<td>4</td>
<td>$\Xi^{(+)}$ (Majorana spinor)</td>
</tr>
<tr>
<td>$[1/2,1/2]^{(-)}$</td>
<td>4</td>
<td>$\Xi^{(-)}$ (Majorana pseudo-spinor)</td>
</tr>
</tbody>
</table>

### A.10. Bosonic 2-forms

<table>
<thead>
<tr>
<th>Representation</th>
<th>Current</th>
<th>Symmetry</th>
<th>Reality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1,1]$</td>
<td>$X_{AB}^{ab} = \overline{\psi}_A \Sigma^{ab} \psi_B$</td>
<td>$X_{AB}^{ab} = X_{AB}^{ab}$ (symm.)</td>
<td>$(X_{AB}^{ab})^* = -X_{AB}^{ab}$ (imag.)</td>
</tr>
<tr>
<td>$[1,0]^{(+)}$</td>
<td>$X_{AB}^{(+a)} = \overline{\psi}_A \wedge \gamma^a \psi_B$</td>
<td>$X_{AB}^{(+a)} = X_{BA}^{(+a)}$ (symm.)</td>
<td>$(X_{AB}^{(+a)})^* = -X_{AB}^{(+a)}$ (imag.)</td>
</tr>
<tr>
<td>$[1,0]^{(-)}$</td>
<td>$X_{AB}^{(-a)} = \overline{\psi}_A \wedge \gamma^a \psi_B$</td>
<td>$X_{AB}^{(-a)} = -X_{BA}^{(-a)}$ (anti-symm.)</td>
<td>$(X_{AB}^{(-a)})^* = X_{AB}^{(-a)}$ (real)</td>
</tr>
<tr>
<td>$[0,0]^{(+)}$</td>
<td>$X_{AB}^{(+)a} = \overline{\psi}_A \wedge \psi_B$</td>
<td>$X_{AB}^{(+)a} = -X_{BA}^{(+)a}$ (anti-symm.)</td>
<td>$(X_{AB}^{(+)a})^* = X_{AB}^{(+)a}$ (real)</td>
</tr>
<tr>
<td>$[0,0]^{(-)}$</td>
<td>$X_{AB}^{(-)a} = \overline{\psi}_A \wedge \gamma^a \psi_B$</td>
<td>$X_{AB}^{(-)a} = -X_{BA}^{(-)a}$ (anti-symm.)</td>
<td>$(X_{AB}^{(-)a})^* = -X_{AB}^{(-)a}$ (imag.)</td>
</tr>
</tbody>
</table>

### A.11. Dimensions of 3-forms in $N$-extended, $D = 4$ superspace

<table>
<thead>
<tr>
<th>$N$</th>
<th>Dim $D(3)$</th>
<th>Dim $\psi^{a,A}$</th>
<th>Dim $(\psi^{a,A} \wedge \psi^{B,B})$</th>
<th>Dim $(\psi^{a,A} \wedge \psi^{B,B} \wedge \psi^{C,C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$44 \oplus 44$</td>
<td>4</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>$148 \oplus 168$</td>
<td>8</td>
<td>36</td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td>$316 \oplus 436$</td>
<td>12</td>
<td>78</td>
<td>364</td>
</tr>
</tbody>
</table>
A.12. Irreducible bases of $D = 4, N = 1$ superspace

$$\psi \wedge \overline{\psi} = \frac{1}{4} \gamma_a X^{(+)a} + \frac{1}{2} \Sigma_{ab} X^{ab}, \ (\Sigma_{ab} = \frac{i}{2} \gamma_{ab}), \ (A.12.1)$$

$$\psi \wedge \overline{\psi} \gamma_a \psi = \psi \wedge X^{(+)a} = -2i \Xi_a^{(12)}, \ (A.12.2)$$

$$\psi \wedge \overline{\psi} \gamma_{ab} \psi = \psi \wedge X_{ab} = \Xi_{ab}^{(8)} - \gamma_{[a} \Xi_{b]}^{(12)}. \ (A.12.3)$$

A.13. Fierz formulas and self-duality identities

$$\psi^\ast = (\frac{1+\gamma_5}{2}) \psi, \ (A.13.1)$$

$$\psi'^\ast = (\frac{1-\gamma_5}{2}) \psi, \ (A.13.2)$$

$$\overline{\psi}^\ast = \overline{\psi} \left(\frac{1+\gamma_5}{2}\right), \ (A.13.3)$$

$$\overline{\psi}'^\ast = \overline{\psi} \left(\frac{1-\gamma_5}{2}\right), \ (A.13.4)$$

$$\gamma_{ab} \psi^\ast \wedge \overline{\psi}'^\ast \wedge \gamma_{ab} \psi'^\ast = 0, \ (A.13.5)$$

$$\gamma_{ab} \psi'^\ast \wedge \overline{\psi}^\ast \wedge \gamma_{ab} \psi^\ast = 0, \ (A.13.6)$$

$$\gamma_{ab} \psi^\ast \wedge \overline{\psi}' \wedge \gamma_{ab} \psi = 0, \ (A.13.7)$$

$$\gamma_{ab} \psi'^\ast \wedge \overline{\psi} \wedge \gamma_{ab} \psi^\ast = 0, \ (A.13.8)$$

$$\gamma_{a} \psi^\ast \wedge \overline{\psi}^\ast \wedge \gamma_{a} \psi^\ast = 0, \ (A.13.9)$$
\[ \gamma^a \psi^* \wedge \overline{\psi}^* \wedge \gamma_a \psi = 0, \quad (A.13.10) \]

\[ \psi^* \wedge \overline{\psi} = \frac{1}{2} \gamma^a \overline{\psi}^* \wedge \gamma_a \psi^*, \quad (A.13.11) \]

\[ \psi_* \wedge \overline{\psi}^* = \frac{1}{2} \gamma^a \overline{\psi}^* \wedge \gamma_a \psi_* , \quad (A.13.12) \]

\[ \psi_* \wedge \overline{\psi} = -\frac{1}{8} \gamma_{ab} \overline{\psi} \wedge \gamma^{ab} \psi_* , \quad (A.13.13) \]

\[ \psi^* \wedge \overline{\psi}^* = -\frac{1}{8} \gamma_{ab} \overline{\psi}^* \wedge \gamma^{ab} \psi^*. \quad (A.13.14) \]

If \( \chi^i \) is a 0-form Majorana spinor with spin 1/2, we have:

\[ \chi^i \overline{\chi} = -\frac{1}{2} \gamma_m \overline{\chi}^i \gamma^m \chi^i, \quad (A.13.15) \]

\[ \chi^i \overline{\chi}^i = -\frac{1}{2} \chi^m \overline{\chi}^i \gamma^m \chi^i + \frac{1}{8} \gamma_{lm} \overline{\chi}^i \gamma^{lm} \chi^i , \quad (A.13.16) \]

\[ \chi^i \overline{\chi} = -\frac{1}{2} \chi^m \overline{\chi} \gamma^m \chi^i, \quad (A.13.17) \]

\[ \chi^i \overline{\chi}^i = -\frac{1}{2} \overline{\chi}^i \chi^i + \frac{1}{8} \gamma_{lm} \overline{\chi} \gamma^{lm} \chi^i . \quad (A.13.18) \]

Table I

**D = 4, N = 1 DE SITTER AND POINCARE' SUPERGRAVITY**

- de Sitter supergravity: \( \overline{\tau} \neq 0 \);

- Poincarè supergravity: \( \overline{\epsilon} = 0 \).

1) **Definition of curvatures**

\[ R^{ab} = d \omega^{ab} - \omega^a \wedge \omega^b + 4 \overline{\epsilon}^2 V^a \wedge V^b + \overline{\epsilon} \overline{\psi} \wedge \gamma^{ab} \psi \]

\[ R^a = \mathcal{D} V^a - \frac{i}{2} \overline{\psi} \wedge \gamma^a \psi \]

\[ \rho = \mathcal{D} \psi - i \overline{\epsilon} \gamma_a \psi \wedge V^a \]
2) Action

\[ A = \int_{M^4 \subset \mathbb{R}^4} R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a \]

3) Inner field equations

\[ R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R_{\mu}^{\nu} = 0 \]

\[ R_{\mu \nu} = 0 \]

\[ \varepsilon_{\mu \nu \alpha} \gamma_5 \gamma_\alpha \rho_{\mu \nu} = 0 \]

4) Bianchi identities

\[ \mathcal{D} R^{ab} - 8 \bar{\psi}^2 R^{[a} \wedge V^{b]} + 2 \bar{\psi} \gamma^{ab} \rho = 0 \]

\[ \mathcal{D} R^a + R^{ab} \wedge V_b - i \bar{\psi} \gamma^a \rho = 0 \]

\[ \mathcal{D} \rho - i \bar{\psi} \gamma_a \psi \wedge R^a - \frac{1}{4} \gamma^{ab} \psi \wedge R^{ab} = 0 \]

5) Field equations in superspace

\[ 2 \varepsilon_{abcd} R^{ab} \wedge V^c + 4 \bar{\psi} \wedge \gamma_5 \gamma_d \rho = 0 \]

\[ 2 \varepsilon_{abcd} R^c \wedge V^d = 0 \]

\[ 8 \gamma_5 \gamma_a \rho \wedge V^a - 4 \gamma_5 \gamma_d \psi \wedge R^a = 0 \]

6) Rheonomic parametrization of curvatures

\[ R^a = 0 \]

\[ R^{ab} = R_{cd}^{ab} V^c \wedge V^d + \bar{\theta}^{ab}_{c} \psi \wedge V^c \]

\[ \rho = \rho_{ab} V^a \wedge V^b \]

\[ \bar{\theta}^{ab}_{c} \equiv \bar{\theta}^{(1)}_{ab} \equiv - \varepsilon^{abxy} \bar{\rho}_{xy} \gamma_5 \gamma_c - \delta^x_c \gamma^{[a} \varepsilon_{b]y} \rho_{xy} \gamma_5 \gamma_m \] (calculated by field equations)

\[ \bar{\theta}^{ab}_{c} \equiv \bar{\theta}^{(2)}_{ab} \equiv - 2 \bar{\rho}_{[a} \gamma^{b]} \gamma_5 \gamma_c - \bar{\theta}^{ab}_{c} \] (calculated by Bianchi identities)

7) Laws of supersymmetry transformation

\[ \delta_\epsilon \omega^{ab} = - 2 \bar{\epsilon} \gamma^{ab} \epsilon + 2 \bar{\theta}^{ab}_{c} \varepsilon V^c \]
\[
\delta_{\epsilon} V^a = i \bar{\psi} \gamma^a \psi
\]
\[
\delta_{\epsilon} \psi = \mathcal{D} \epsilon - i \bar{\psi} \gamma_a \epsilon V^a
\]

We have:

a) symmetries (closed algebra) of inner equations 3);

b) invariances (on-shell closed algebra, off-shell open algebra) of the action 2) in the first order formalism if \( \bar{\theta}^{a b} = \bar{\theta}^{(I) a b} \), in the second order if \( \bar{\theta}^{a b} = \bar{\theta}^{(II) a b} \).

Table II

**D = 4, N = 1 Supergravity Coupled to n Scalar Multiplets**

1) **Definition of curvatures**

\[
R^a = \mathcal{D} V^a - i \bar{\psi} \gamma^a \psi. \quad (\mathcal{D} V^a = dV^a - \omega^{ab} \wedge V_b)
\]

\[
R^{ab} = d\omega^{ab} - \omega^{ae} \wedge \omega_e^b
\]

\[
\rho_* = \nabla \psi_* = d\psi_* - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab} \psi_* + i \frac{1}{2} Q \wedge \psi_*
\]

\[
R (z)^i = dz_i = dz^i
\]

\[
R (\chi)^i = \nabla \chi^j = d\chi^i - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab} \chi^i + \left\{ \frac{1}{2} \right\} d\chi^j \chi^k - i \frac{1}{2} Q \chi^i
\]

\[
Q = \frac{1}{2i} (\partial_j G dz^i - \partial_i G dz^j)
\]

2) **Parametrization of curvatures**

\[
R^a = 0
\]

\[
R^{ab} = R_{cd}^{ab} V^c \wedge V^d - (2i \bar{\psi}^* \gamma^a \rho_*^{b} \gamma^{cd} + 2i \bar{\psi}^* \gamma^a \rho^{b} \gamma^{cd} - i \bar{\psi}^* \gamma^c \rho_*^{ab} - i \bar{\psi}^* \gamma^c \rho^{ab} \gamma^{cd} \psi_* - i S \bar{\psi}^* \gamma^{ab} \psi_* - i \frac{4}{3} T \bar{\psi}^* \gamma_d \psi_* \epsilon^{abcd} +
\]

-68-
\[ \rho_\ast = \rho_{ab} V_a \wedge V_b + \frac{i}{8} T_a \gamma^{ab} \psi_\ast \wedge V_\ast + S \gamma_a \psi_\ast \wedge V^a \]

\[ dz^j = Z^j_a V^a + \overline{\psi} \psi_\ast. \]

\[ \nabla \chi' = \nabla_a \chi^j V^a + i Z^j_a \gamma^a \psi_\ast + \mathcal{K} \ast_i \psi_\ast. \]

with:

\[ T_a = g_{ab} \overline{\chi} \gamma_a \chi^j. \]

\[ S = i e \exp\left( \frac{G}{2} \right) \]

\[ \mathcal{K} \ast_i = 2 e (g^{ab} \partial_j G) \exp\left( \frac{G}{2} \right) \]

3) Laws of supersymmetry transformation

\[ \delta V^a = i \bar{\psi} \gamma^a \psi_\ast - i \bar{\psi} \gamma^a \epsilon_\ast. \]

\[ \delta \psi_\ast = \nabla \epsilon_\ast + \frac{i}{8} T_a \gamma^{ab} \epsilon_\ast V_b + S \gamma_a \epsilon_\ast V^a + \frac{1}{4} (\partial_j G \overline{\chi} \epsilon_\ast - \partial_j G \overline{\chi} \epsilon_\ast \psi_\ast) \psi_\ast. \]

\[ \delta z^j = \overline{\chi} \epsilon_\ast. \]

\[ \delta \chi' = i Z^j_a \gamma^a \epsilon_\ast + \mathcal{K} \ast_i \epsilon_\ast - \frac{1}{4} (\partial_j G \overline{\chi} \epsilon_\ast - \partial_j G \overline{\chi} \epsilon_\ast \psi_\ast) \chi^j \]

with:

\[ \nabla \epsilon_\ast = \mathcal{D} \epsilon_\ast + \frac{1}{4} Q \epsilon_\ast. \]

4) Complete Lagrangian

\[ \mathcal{L} \text{ (SUGRA+WZ)} = \epsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4 (\overline{\psi}^\ast \wedge \gamma_a \rho_\ast + \overline{\rho}^\ast \wedge \gamma_a \psi_\ast) \wedge V^a - \frac{i}{3} g_{j} (\overline{\chi}^{j} \gamma_a \nabla \overline{\chi} \gamma_a \nabla \chi^j) \wedge V_b \wedge V_c \wedge V_d \epsilon_{abcd} + \]

\[ + \frac{2}{3} g_{j} (Z^j_a (d\overline{z} - \overline{X} \psi^\ast)) + \overline{Z}^j_a (d\overline{z} - \overline{X} \psi_\ast)) \wedge V_b \wedge V_c \wedge V_d \epsilon_{abcd} - \]

\[ - \frac{1}{6} g_{j} Z^j_a Z^{j''} \epsilon_{b_1 b_2 b_3} V^b \wedge V^b \wedge V^b \wedge V^b - \]

-69-
\[-2 \, g_{ij} \, (dz^a \wedge \bar{\chi}^i \, \gamma_{ab} \, \psi^* - d \bar{\chi}^i \wedge \bar{\chi}^i \, \gamma_{ab} \, \psi^*) \wedge V^a \wedge V^b -
\]
\[-R^a \wedge V_a \wedge g_{ij} \, \bar{\chi}^i \, \gamma_{ab} \, \chi^j \wedge V^b + 2 \, g_{ij} \, \bar{\chi}^i \, \gamma_{ab} \, \chi^j \wedge \gamma_{ab} \, \psi^* \wedge V^a \wedge V^b -
\]
\[-\frac{1}{48} \, e_{abcd} \wedge V^a \wedge V^b \wedge V^c \wedge V^d \, \bar{\chi}^i \, \gamma_{ab} \, \chi^j \wedge \bar{\chi}^i \, \gamma_{ab} \, \chi^j \wedge (g_{ij} \, g_{kl} + R_{ijkl}) -
\]
\[-4 \, (S \, \bar{\psi}^* \wedge \gamma_{ab} \, \psi^* + S^* \, \psi^* \wedge \gamma_{ab} \, \psi^*) \wedge V^a \wedge V^b + (\bar{\psi} \, i \, \bar{\chi}^i \, \gamma_a \, \psi^* + \bar{\psi} \, i \, \bar{\chi}^i \, \gamma_a \, \psi^*) \wedge V_b \wedge V_c \wedge V_d \, e_{abcd} + (m_g \, \bar{\chi}^i \, \chi^j + m_g \, \bar{\chi}^i \, \chi^j + W) \, e_{abcd} \wedge V^a \wedge V^b \wedge V^c \wedge V^d
\]

with:

\[S = i \, e \, \exp \left[ \frac{1}{2} \, G(z, \bar{z}) \right] \]

\[F_{\, i} = \frac{4}{3} i \, e \, \delta_i \, G \, \exp \left[ \frac{1}{2} \, G(z, \bar{z}) \right] \]

\[m_g = \frac{e}{6} \, (\partial_i \, G \, \partial_j \, G + \nabla_i \, \partial_j \, G) \, \exp \left[ \frac{1}{2} \, G(z, \bar{z}) \right] \]

\[W = -\frac{2}{3} \, \nu^2 \, (3 - g^{ij} \, \partial_i \, G \, \partial_j \, G) \, \exp \left[ \frac{1}{2} \, G(z, \bar{z}) \right] \]
BIOGRAPHY

Paolo Di Sia is currently adjunct professor by the University of Padova (Italy). He obtained a bachelor in metaphysics, a master in theoretical physics and a PhD in theoretical physics applied to nanotechnology. He interested in classical-quantum-relativistic nanophysics, theoretical physics, Planck scale physics, metaphysics, mind-brain science, history and philosophy of science, science education. He is author of 234 works to date (papers on national and international journals, international book chapters, books, internal academic notes, works on scientific web-pages, popular works, in press), is reviewer of two mathematics academic books, reviewer of 12 international journals. He obtained 9 international awards, has been included in Who’s Who in the World every year since 2015, selected for 2017 “Albert Nelson Marquis Lifetime Achievement Award”, is member of 7 scientific societies and of 30 International Advisory/Editorial Boards.

http://www.paolodisia.com