Available online at www.worldscientificnews.com

World Scientific News

WSN 90 (2017) 77-87 EISSN 2392-2192

Performance Measure of a New One-Step Numerical Technique via Interpolating Function for the Solution of Initial Value Problem of First Order Differential Equation

Fadugba Sunday Emmanuel[1,*] and **Okunlola Joseph Temitayo**[2]

[1]Department of Mathematics, Ekiti State University, Ado Ekiti, Nigeria

[2]Department of Mathematical and Physical Sciences, Afe Babalola University, Ado Ekiti, Nigeria

E-mail address: emmasfad2006@yahoo.com

ABSTRACT

This paper presents the development of a new one-step numerical technique for the solution of initial value problems of first order differential equations by means of the interpolating function. The interpolating function used in this paper consists of both polynomial and exponential functions. Numerical experiments were performed to determine the efficiency and robustness of the scheme. The results show that the scheme is computationally efficient, robust and compares favourably with exact solutions.

***Keywords*:** Initial value problem, Interpolating function, One-step numerical technique

2010 Mathematics Subject Classification: 45L05, 49K15, 65L05

1. INTRODUCTION

In sciences and engineering, mathematical models are formulated to aid in the understanding of physical phenomena. The formulated model often yields an equation that

(Received 08 Nowember 2017; Accepted 26 November 2017; Date of Publication 27 November 2017)

contains the derivatives of an unknown function. Such an equation is referred to as Differential equation. Interestingly, differential equations arising from the modeling of physical phenomena often do not have exact solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary.

We consider the first order ordinary differential equation of the form

$$y' = f(x, y) \tag{1}$$

with the associated condition

$$y(x_0) = y_0 \tag{2}$$

for $a \leq x \leq b, -\infty < y < \infty$

The combination of (1) and (2) is called initial value problem written as

$$y'(x) = f(x, y),\ y(x_0) = y_0,\ x \in [a, b],\ y \in (-\infty, \infty) \tag{3}$$

We assume that *f(x)* satisfies Lipschitz condition which guarantees the existence and uniqueness of solution of (3).

Many numerical analysts have derived several schemes for the solution of the initial value problems in ordinary differential equations of the form (3).

The single-step methods include those developed by Fadugba and Falodun (2017), Ayinde and Ibijola (2015), Fatunla (1976), Kama and Ibijola (2000), Lambert (1991), just to mention a few.

These methods were constructed by representing the theoretical solution *y(x)* to (3) in the interval, $[x_n, x_{n+1}]$, $[n \geq 0]$ by linear and non-linear polynomial interpolating functions. Also on the other hand, authors like Butcher (2003), Zarina et al. (2005), Awoyemi et al. (2007), Areo et al. (2011), Ibijola, Skwame and Kumleng (2011) have all proposed linear multistep methods (LMMs) to generate numerical solution to (3). These authors proposed methods in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials.

In this paper we develop a new scheme to solve the initial value problem (3). The rest of the paper is outlined as follows: Section Two is the development of the new one-step numerical technique via the interpolating function. Section Three consists of the implementation of the technique. Section Four consists of discussion of results and concluding remarks.

2. DEVELOPMENT OF A NEW ONE-STEP TECHNIQUE VIA NTERPOLATING FUNCTION

Ordinary differential equations (ODEs) arise from mathematical modeling of human activities in sciences, engineering, control theory, optimization, management and technology. Let us assume that the exact solution to (3) is given by $y(x)$. Consider an interpolating function of the form

$$F(x) = (\alpha_1 + \alpha_2 + \alpha_3)e^{2x} + \alpha_4 x + \alpha_5 x^2 + \alpha_6 \quad (4)$$

with integration interval of $[a, b]$ in the form

$$a = x_0 < \cdots < x_{n+1} < \cdots < x_N = b$$

with step size

$$h = x_{n+1} - x_n, \ \ n = 0,1,2, \ldots \quad (5)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are real undetermined coefficients and α_6 is a constant.

The mesh point is defined as

$$\left.\begin{array}{l} x_n = x_0 + nh, n = 1,2, \ldots \\ \text{or} \\ x_{n+1} = x_0 + (n+1)h, n = 0,1,2, \ldots \end{array}\right\} \quad (6)$$

Expanding (4) at x_n and x_{n+1}, we have respectively

$$F(x_n) = (\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} + \alpha_4 x_n + \alpha_5 x_n^2 + \alpha_6 \quad (7)$$

and

$$F(x_{n+1}) = (\alpha_1 + \alpha_2 + \alpha_3)e^{2x_{n+1}} + \alpha_4 x_{n+1} + \alpha_5 x_{n+1}^2 + \alpha_6 \quad (8)$$

Differentiating (7), we set the $1^{st}, 2^{nd}$ and 3^{rd} derivatives as

$$F'(x_n) = 2(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} + \alpha_4 + 2\alpha_5 x_n \quad (9)$$

$$F''(x_n) = 4(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} + 2\alpha_5 \quad (10)$$

$$F'''(x_n) = 8(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} \quad (11)$$

The derivatives (9), (10) and (11) are required to be equal to the following identities $f_n, f_n^{(1)}$ and $f_n^{(2)}$ respectively.

Therefore,

$$F'(x_n) = f_n \quad (12)$$

$$F''(x_n) = f_n^{(1)} \tag{13}$$

$$F'''(x_n) = f_n^{(2)} \tag{14}$$

This implies that

$$2(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} + 2\alpha_5 x_n = f_n \tag{15}$$

$$4(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} + 2\alpha_5 = f_n^{(1)} \tag{16}$$

$$8(\alpha_1 + \alpha_2 + \alpha_3)e^{2x_n} = f_n^{(2)} \tag{17}$$

From (17)

$$(\alpha_1 + \alpha_2 + \alpha_3) = \frac{f_n^{(2)}}{8e^{2x_n}} \tag{18}$$

Substituting (18) into (16), yields

$$4\left(\frac{f_n^{(2)}}{8e^{2x_n}}\right)e^{2x_n} + 2\alpha_5 = f_n^{(1)}$$

$$\frac{1}{2}f_n^{(2)} + 2\alpha_5 = f_n^{(1)}$$

$$2\alpha_5 = f_n^{(1)} - \frac{1}{2}f_n^{(2)}$$

$$\alpha_5 = \frac{1}{2}\left(f_n^{(1)} - \frac{1}{2}f_n^{(2)}\right) = \frac{1}{2}f_n^{(1)} - \frac{1}{4}f_n^{(2)} \tag{19}$$

Substituting (18) and (19) into (15)

$$2\left(\frac{f_n^{(2)}}{8e^{2x_n}}\right)e^{2x_n} + \alpha_4 + 2\left(\frac{1}{2}f_n^{(1)} - \frac{1}{4}f_n^{(2)}\right)x_n = f_n$$

$$\frac{1}{4}f_n^{(2)} + \alpha_4 + \left(f_n^{(1)} - \frac{1}{2}f_n^{(2)}\right)x_n = f_n$$

$$\alpha_4 = \left(f_n - \frac{1}{4}f_n^{(2)}\right) - \left(f_n^{(1)} - \frac{1}{2}f_n^{(2)}\right)x_n \tag{20}$$

Thus, we have the following

$$\left.\begin{aligned}
&(\alpha_1 + \alpha_2 + \alpha_3) = \left(\frac{f_n^{(2)}}{8e^{2x_n}}\right)\\
&\alpha_4 = \left(f_n - \frac{1}{4}f_n^{(2)}\right) - \left(f_n^{(1)} - \frac{1}{2}f_n^{(2)}\right)x_n\\
&\alpha_5 = \frac{1}{2}f_n^{(1)} - \frac{1}{4}f_n^{(2)}
\end{aligned}\right\} \tag{21}$$

Using the fact that

$$\left.\begin{aligned}
&F(x_n) = y(x_n)\\
&F(x_{n+1}) = y(x_{n+1})
\end{aligned}\right\} \tag{22}$$

We can write,

$$\left.\begin{aligned} y(x_{n+1}) &= y_{n+1} \\ y(x_n) &= y_n \end{aligned}\right\} \tag{23}$$

where: y_n is the numerical solution to the initial value problem given by (3). From (22) and (23), we write

$$y_{n+1} - y_n = F(x_{n+1}) - F(x_n) \tag{24}$$

We first expand the RHS of (24)

$$F(x_{n+1}) - F(x_n) = (\alpha_1 + \alpha_2 + \alpha_3)(e^{2x_{n+1}} - e^{2x_n}) + \alpha_4(x_{n+1} - x_n) + \alpha_5(x_{n+1}^2 - x_n^2) \tag{25}$$

From (6), we have that

$x_n = x_0 + nh$ and $x_{n+1} = x_0 + (n+1)h$

Therefore,

$$x_{n+1} - x_n = h$$

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= (x_0 + (n+1)h^2) - (x_0 + nh)^2 \\ &= x_0^2 + 2x_0(n+1)h + (n+1)^2h^2 - x_0^2 - 2x_0nh - n^2h^2 \\ &= 2x_0nh + 2x_0h + (n^2 + 2n + 1)h^2 - 2x_0nh - n^2h^2 \\ &= 2x_0h + n^2h^2 + 2nh^2 + h^2 - n^2h^2 \\ &= 2x_0h + h^2(2n+1) \end{aligned} \tag{26}$$

It is worth mentioning here that x_0 varies which makes our scheme an avenue to solve any problem whose initial condition is not only limited to $y(0) = 1$

Setting $x_0 = 0$ in (26), yields

$$x_{n+1}^2 - x_n^2 = h^2\,(2n+1)$$

Substituting (5) and (26) into (25), we have that

$$\begin{aligned} F(x_{n+1}) - F(x_n) &= (\alpha_1 + \alpha_2 + \alpha_3)\left(e^{2(n+1)h} - e^{2nh}\right) + \alpha_4(h) + \alpha_5(h^2(2n+1)) \\ &= (\alpha_1 + \alpha_2 + \alpha_3)e^{2nh}(e^{2h} - 1) + \alpha_4h + \alpha_5h^2(2n+1) \end{aligned} \tag{27}$$

Substituting (21) into (27) with $x_n = nh$ and $x_{n+1} = (n+1)h$

$$F(x_{n+1}) - F(x_n) = \frac{f_n^{(2)}}{8}(e^{2h} - 1) + \left[\left(f_n - \frac{1}{4} f_n^{(2)}\right) - \left(f_n^{(1)} - \frac{1}{2} f_n^{(2)}\right) nh\right] h$$

$$+ \left(\frac{1}{2} f_n^{(1)} - \frac{1}{4} f_n^{(2)}\right) h^2 (2n + 1) \qquad (28)$$

$$= \frac{f_n^{(2)}}{8}(e^{2h} - 1) + \left(f_n - \frac{1}{4} f_n^{(2)}\right) h + \left[\left(\frac{1}{2} f_n^{(1)} - \frac{1}{4} f_n^{(2)}\right)(2n + 1) - n\left(f_n^{(1)} - \frac{1}{2} f_n^{(2)}\right)\right] h^2$$

Substituting (28) into (24), we get

$$y_{n+1} = y_n + \frac{f_n^{(2)}}{8}(e^{2h} - 1) + \left(f_n - \frac{1}{4} f_n^{(2)}\right) h$$

$$+ \left(\left(\frac{1}{2} f_n^{(1)} - \frac{1}{4} f_n^{(2)}\right)(2n + 1) - \left(f_n^{(1)} - \frac{1}{2} f_n^{(2)}\right) n\right) h^2 \qquad (29)$$

Equation (29) is the new one-step numerical technique.

3. IMPLEMENTATION OF THE NEW ONE-STEP TECHNIQUE

This section presents some numerical experiments as follows.

3. 1. NUMERICAL EXPERIMENTS

It is always necessary to demonstrate the applicability, suitability and accuracy of the newly developed one-step numerical method. To do this, the method was rewritten in an algorithm form, translated into computer codes using MATLAB programming language and implemented with sample problems on a digital computer. We consider the following numerical experiments.

3. 1. 1. EXPERIMENT 1

Consider the initial value problem of the form:

$y' = 2xy, \quad y(0) = 1, \;\; 0 \le x \le 1$

with step size $h = 0.1$, whose exact/theoretical solution is given by

$y(x) = e^{x^2}$

The comparative analyzes of the results are displayed in Table 1 below.

Table 1. The comparative analysis of the results generated via the new technique ('YN') in the context of the exact solution ('YXN')

'N'	'XN'	'YN'	'YXN'	'EN'
0.00	0.0000000000	1.0000000000	1.0000000000	0.0000000000
1.00	0.1000000000	1.0100000000	1.0100501671	0.0000501671
2.00	0.2000000000	1.0407159346	1.0408107742	0.0000948395
3.00	0.3000000000	1.0940339449	1.0941742837	0.0001403388
4.00	0.4000000000	1.1733176174	1.1735108710	0.0001932536
5.00	0.5000000000	1.2837636842	1.2840254167	0.0002617324
6.00	0.6000000000	1.4329722167	1.4333294146	0.0003571979
7.00	0.7000000000	1.6318192914	1.6323162200	0.0004969285
8.00	0.8000000000	1.8957726578	1.8964808793	0.0007082215
9.00	0.9000000000	2.2468726394	2.2479079867	0.0010353473
10.00	1.0000000000	2.7167304092	2.7182818285	0.0015514193

3. 1. 2. EXPERIMENT 2

Consider the initial value problem of the form:

$$y' = y, \quad y(0) = 1, \quad 0 \leq x \leq 1$$

with step size $h = 0.1$, whose exact solution is given by

$$y(x) = e^x$$

The comparative analyzes of the result are displayed in Table 2 below.

Table 2. The comparative analysis of the results generated via the new technique ('YN') in the context of the exact solution ('YXN')

'N'	'XN'	'YN'	'YXN'	'EN'
0.00	0.0000000000	1.0000000000	1.0000000000	0.0000000000
1.00	0.1000000000	1.1051753448	1.1051709181	0.0000044267
2.00	0.2000000000	1.2214125427	1.2214027582	0.0000097845

3.00	0.3000000000	1.3498750280	1.3498588076	0.0000162204
4.00	0.4000000000	1.4918485994	1.4918246976	0.0000239018
5.00	0.5000000000	1.6487542902	1.6487212707	0.0000330195
6.00	0.6000000000	1.8221625911	1.8221188004	0.0000437907
7.00	0.7000000000	2.0138091699	2.0137527075	0.0000564624
8.00	0.8000000000	2.2256122436	2.2255409285	0.0000713151
9.00	0.9000000000	2.4596917787	2.4596031112	0.0000886675
10.00	1.0000000000	2.7183907095	2.7182818285	0.0001088811

3. 1. 3. EXPERIMENT 3

Consider the initial value problem of the form:

$$y' = 1 + y^2, \;\; y(0) = 1, \;\; 0 \leq x \leq 0.5$$

with step size $h = 0.01$, whose exact solution is given by

$$y(x) = tan\left(x + \frac{\pi}{4}\right)$$

The comparative analyzes of the results are displayed in Table 3 below

Table 3. The comparative analysis of the results generated via the new technique ('YN') in the context of the exact solution ('YXN')

'N'	'XN'	'YN'	'YXN'	'EN'
0.00	0.0000000000	1.0000000000	1.0000000000	0.0000000000
1.00	0.0100000000	1.0202026801	1.0202027004	0.0000000204
2.00	0.0200000000	1.0408218379	1.0408218807	0.0000000427
3.00	0.0300000000	1.0618747405	1.0618748078	0.0000000673
4.00	0.0400000000	1.0833795661	1.0833796603	0.0000000942
5.00	0.0500000000	1.1053554667	1.1053555905	0.0000001238
6.00	0.0600000000	1.1278226354	1.1278227917	0.0000001563
7.00	0.0700000000	1.1508023794	1.1508025714	0.0000001919
8.00	0.0800000000	1.1743171993	1.1743174304	0.0000002311

9.00	0.0900000000	1.1983908748	1.1983911490	0.0000002742
10.00	0.1000000000	1.2230485589	1.2230488804	0.0000003216
11.00	0.1100000000	1.2483168792	1.2483172529	0.0000003737
12.00	0.1200000000	1.2742240495	1.2742244805	0.0000004311
13.00	0.1300000000	1.3007999904	1.3008004846	0.0000004943
14.00	0.1400000000	1.3280764623	1.3280770262	0.0000005639
15.00	0.1500000000	1.3560872104	1.3560878511	0.0000006408
16.00	0.1600000000	1.3848681231	1.3848688487	0.0000007256
17.00	0.1700000000	1.4144574070	1.4144582264	0.0000008194
18.00	0.1800000000	1.4448957782	1.4448967013	0.0000009231
19.00	0.1900000000	1.4762266733	1.4762277112	0.0000010379
20.00	0.2000000000	1.5084964820	1.5084976471	0.0000011651
21.00	0.2100000000	1.5417548042	1.5417561104	0.0000013062
22.00	0.2200000000	1.5760547336	1.5760561964	0.0000014629
23.00	0.2300000000	1.6114531727	1.6114548098	0.0000016371
24.00	0.2400000000	1.6480111824	1.6480130134	0.0000018310
25.00	0.2500000000	1.6857943699	1.6857964172	0.0000020472
26.00	0.2600000000	1.7248733225	1.7248756111	0.0000022886
27.00	0.2700000000	1.7653240899	1.7653266483	0.0000025584
28.00	0.2800000000	1.8072287254	1.8072315859	0.0000028605
29.00	0.2900000000	1.8506758919	1.8506790912	0.0000031993
30.00	0.3000000000	1.8957615429	1.8957651229	0.0000035799
31.00	0.3100000000	1.9425896893	1.9425936976	0.0000040083
32.00	0.3200000000	1.9912732653	1.9912777566	0.0000044913
33.00	0.3300000000	2.0419351083	2.0419401453	0.0000050370
34.00	0.3400000000	2.0947090706	2.0947147254	0.0000056548
35.00	0.3500000000	2.1497412844	2.1497476402	0.0000063558

36.00	0.3600000000	2.2071916054	2.2071987585	0.0000071531
37.00	0.3700000000	2.2672352645	2.2672433265	0.0000080620
38.00	0.3800000000	2.3300647642	2.3300738651	0.0000091009
39.00	0.3900000000	2.3958920624	2.3959023540	0.0000102915
40.00	0.4000000000	2.4649510966	2.4649627567	0.0000116601
41.00	0.4100000000	2.5375007110	2.5375139489	0.0000132379
42.00	0.4200000000	2.6138280645	2.6138431272	0.0000150626
43.00	0.4300000000	2.6942526137	2.6942697939	0.0000171803
44.00	0.4400000000	2.7791307874	2.7791504340	0.0000196466
45.00	0.4500000000	2.8688614981	2.8688840280	0.0000225299
46.00	0.4600000000	2.9638926682	2.9639185827	0.0000259145
47.00	0.4700000000	3.0647289982	3.0647589027	0.0000299045
48.00	0.4800000000	3.1719412601	3.1719758901	0.0000346300
49.00	0.4900000000	3.2861774784	3.2862177323	0.0000402540
50.00	0.5000000000	3.4081764599	3.4082234423	0.0000469824

4. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

In this paper, we have proposed a new one step numerical technique for the solution of first order ordinary differential equations. Some numerical experiments were performed. Numerical results were also compared with the exact solutions which clearly showed the efficiency of the new method. The mesh points (**'XN'**), numerical solution (**'YN'**), the exact solution (**'YXN'**) and the absolute errors (**'EN'**) are displayed in second, third, fourth and fifth columns respectively. It is observed from Tables 1, 2 and 3 above that the newly proposed technique is efficient, robust, accurate and very close to the exact solution.

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