



SHORT COMMUNICATION

## On the old and new matrix representations of the Clifford algebra for the Dirac equation and quantum field theory

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### ABSTRACT

The standard 16-dimensional and new 64-dimensional representations of the Clifford algebras in the terms of Dirac  $\gamma$  matrices are under consideration. The matrix 64-dimensional representation of the Clifford algebra  $Cl^{\mathbb{R}}(0,6)$  over the field of real numbers is presented. The relationship of this representation to the matrix representation of 28-dimensional  $SO(8)$  algebra, which contains the standard and additional spin operators, is given. The role of matrix representations of the Clifford algebra in the quantum field theory is described. The role of matrix representations of  $Cl^{\mathbb{R}}(0,6)$  and  $SO(8)$  algebras in the proof of Fermion-Boson duality property of the Dirac and higher spin Dirac-like equations is demonstrated.

**Keywords:** Clifford algebra, spinor field, Dirac equation, arbitrary spin

## 1. INTRODUCTION

The matrix representations of the Clifford algebra (on the Clifford algebra see, e.g., [1, 2]) have the important applications in modern theoretical physics. First, these mathematical objects are useful in the relativistic quantum mechanics and field theory.

Note that one of the principal objects of these contemporary models of physical reality is the Dirac equation, see, e.g. [3]. This equation describes the particle-antiparticle doublet with spins  $s = (1/2, 1/2)$  or, in other words, spin  $1/2$  fermion-antifermion doublet. The presentation of Dirac equation operator  $D \equiv i\gamma^\mu \partial_\mu - m$ , as well as a list of other operators of quantum spinor field, in the terms of a  $\gamma$  matrices enable one to use the anti-commutation relations between the Clifford algebra operators directly for finding the symmetries, solutions, conservation laws, fulfilling the canonical quantization and calculating the interaction processes in the quantum field models. The important fact is that application of the Clifford algebra essentially simplifies the calculations.

Thus, for the case of free non-interacting spinor field this equation has the form

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \quad (1)$$

where

$$x \in M(1,3), \quad \partial_\mu \equiv \partial/\partial x^\mu, \quad \mu = \overline{0,3}, \quad (2)$$

$M(1,3)$  is the Minkowski space-time and 4-component function  $\psi(x)$  belongs to rigged Hilbert space

$$S^{3,4} \subset H^{3,4} \subset S^{3,4*}. \quad (3)$$

Note that due to a special role of the time variable  $x^0 = t \in (x^\mu)$  (in obvious analogy with nonrelativistic theory), in general consideration one can use the quantum-mechanical rigged Hilbert space (3). Here the Schwartz test function space  $S^{3,4}$  is dense in the Schwartz generalized function space  $S^{3,4*}$  and  $H^{3,4}$  is the quantum-mechanical Hilbert space of 4-component functions over  $\mathbb{R}^3 \subset M(1,3)$ .

In order to finish with notations, assumptions and definitions let us note that here the system of units  $\hbar = c = 1$  is chosen, the metric tensor in Minkowski space-time  $M(1,3)$  is given by

$$g = (g_\nu^\mu) = \text{diag } g(+1, -1, -1, -1), \quad g^{\mu\nu} = g_{\mu\nu} = g_\nu^\mu, \quad x_\mu = g_{\mu\nu} x^\nu, \quad (4)$$

and summation over the twice repeated indices is implied. The Dirac  $\gamma$  matrices are taken in the standard Dirac-Pauli representation

$$\gamma^0 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \gamma^l = \begin{vmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{vmatrix}, \quad l = 1, 2, 3, \quad (5)$$

where the Pauli matrices are given by

$$\sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma^2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \sigma^1 \sigma^2 = i \sigma^3, \text{ 123!-circle.} \quad (6)$$

**2. STANDARD MATRIX REPRESENTATIONS OF THE  $Cl^C(1,3)$  CLIFFORD AND THE  $SO(1,5)$  LIE ALGEBRS**

Four operators (5) satisfy the anti-commutation relations of the Clifford algebra

$$\begin{aligned} \gamma^\mu (\mu = \overline{0,3}) : \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2g^{\mu\nu}, \\ \gamma_\mu &= g_{\mu\nu} \gamma^\nu, \gamma^{\dagger k} = -\gamma^k, \gamma^{\dagger 0} = \gamma^0, \end{aligned} \quad (7)$$

and realize the 16-dimensional ( $2^4 = 16$ ) matrix representation of the Clifford algebra  $Cl^C(1,3)$  over the field of complex numbers.

For our purposes we introduce the additional matrix

$$\gamma^4 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \begin{vmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{vmatrix}, \mathbf{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \quad (8)$$

Such matrix satisfies the anti-commutation relations of the Clifford algebra as well

$$\gamma^{\bar{\mu}} \gamma^{\bar{\nu}} + \gamma^{\bar{\nu}} \gamma^{\bar{\mu}} = 2g^{\bar{\mu}\bar{\nu}}; \bar{\mu}, \bar{\nu} = \overline{0,4}, (g^{\bar{\mu}\bar{\nu}}) = (+----), \quad (9)$$

where metric tensor is given by  $5 \times 5$  matrix.

**Table 1.** Elements of the  $\gamma$  matrix representation of the  $Cl^C(1,3)$  algebra

$\mathbf{I}_4 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4$					
	$\gamma^0 \gamma^1$	$\gamma^0 \gamma^2$	$\gamma^0 \gamma^3$	$\gamma^0 \gamma^4$	$\gamma^0 \gamma^5 \equiv \gamma^0$
		$\gamma^1 \gamma^2$	$\gamma^1 \gamma^3$	$\gamma^1 \gamma^4$	$\gamma^1 \gamma^5 \equiv \gamma^1$
			$\gamma^2 \gamma^3$	$\gamma^2 \gamma^4$	$\gamma^2 \gamma^5 \equiv \gamma^2$
				$\gamma^3 \gamma^4$	$\gamma^3 \gamma^5 \equiv \gamma^3$
					$\gamma^4 \gamma^5 \equiv \gamma^4$

In the papers [4,5] 16 elements of the Clifford-Dirac algebra are linked to the 15 elements of the SO(3,3) Lie algebra. In our papers [6-12] we were able to give the link between the  $\gamma$  matrix representation of the  $Cl^C(1,3)$  algebra (Table 1) and the 15 elements of the SO(1,5) Lie algebra:

$$s^{\bar{\mu}\bar{\nu}} \equiv \frac{1}{4}[\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}], \quad s^{\bar{\mu}5} = -s^{5\bar{\mu}} = \frac{1}{2}\gamma^{\bar{\mu}}, \quad \bar{\mu}, \bar{\nu} = \overline{0,4}. \quad (10)$$

The generators (10) satisfy the commutation relations of the SO(1,5) Lie algebra.

Thus, it is proved that one and the same 15 not unit elements determine both the matrix representation of the  $Cl^C(1,3)$  Clifford algebra and the  $\gamma$  matrix representation of the SO(1,5) Lie algebra (similarly for SO(6) algebra). Note that elements of the SO(1,5) representation (10) contain the spin operators of the spinor field and Dirac theory. Note that here in (10) the anti-Hermitian realization of the SO(1,5) operators is chosen, for the reasons see, e.g., [6-12].

### **3. MATRIX REPRESENTATIONS OF THE $Cl^R(0,6)$ CLIFFORD AND THE SO(8) LIE ALGEBRS**

In addition to the Dirac equation the  $\gamma$  matrix representations of the Clifford algebra are used widely for the multicomponent Dirac-like equations of arbitrary spin such as the Bhabha [13], Bargman-Wigner [14], Rarita-Schwinger (for the field with the spin  $s = 3/2$  [15]), Iwanenko-Landau-Dirac-Kähler (see, e.g., [16]) equations. Note that these equations can describe both fermions and bosons. Moreover, new equations for an arbitrary spin [17-21], which are derived from the relativistic canonical quantum mechanics, are based on the  $\gamma$  matrix representations of the Clifford algebra as well. Therefore, the application of the Clifford algebra in the quantum theory is more wide then the 4-component Dirac equation and corresponded spinor field. It is evident that the program of finding of more wide matrix representations then the 16-dimensional  $Cl^C(1,3)$  and 15-dimensional SO(1,5) algebras representations is the interesting task. The corresponding matrix representations can be useful both in the known quantum field theory models and in the development of new approaches.

This problem has been considered in [6-12], where the new 64-dimensional matrix representations of the  $Cl^R(0,6)$  and  $Cl^R(4,2)$  Clifford algebras have been found as well as the 28-dimensional representation of the SO(8) Lie algebra, all over the field of real numbers. Contrary to the well-known matrix representation of the SO(1,5) algebra (15 generators of which accurate to the  $1/2$  coefficient coincide with 15 not unit elements of the standard  $\gamma$  matrix representation of the  $Cl^C(1,3)$  Clifford algebra and, furthermore, determine the 16-dimensional representation of this Clifford algebra as well) 64-dimensional matrix representations of the  $Cl^R(0,6)$  and  $Cl^R(4,2)$  Clifford algebras essentially do not coincide with 28-dimensional  $\gamma$  matrix representation of the SO(8) Lie algebra. The 28 generators of the SO(8) algebra  $\gamma$  matrix representation do not realize the Clifford algebra even after

multiplication by factor 2 and adding the unit element. In order to find some  $\gamma$  matrix representation of the Clifford algebra on the basis of 28 SO(8) generators one must essentially expand the list of these operators. Thus, one comes to the 64-dimensional matrix representations of  $C\ell^{\mathbb{R}}(0,6)$  and  $C\ell^{\mathbb{R}}(4,2)$  Clifford algebras over the field of real numbers, which have been found earlier in [6-12].

By taking into account these new  $\gamma$  matrix representations of the Clifford and SO(8) algebras the Bose symmetries, solutions and conservation laws for the standard Dirac equation and spinor field with nonzero mass have been found [6-12]. Therefore, the Fermi-Bose duality of this equation has been proved. Furthermore, in [17-21] the Fermi-Bose duality of the multicomponent Dirac-like equations for higher spin has been proved as well.

Let us consider briefly the new matrix representations of the Clifford and SO(8) algebras and the history how these algebras have been put into consideration. We have started from the case of *massless Dirac equation* and the *spinor field of zero mass*. Indeed, at first the Bose symmetries of the massless Dirac equation have been found. We have used essentially so-called Pauli-Gürsey-Ibragimov symmetry [22, 23] of the Dirac equation with zero mass, i. e. the fact that this equation is invariant with respect to the transformations generating by the eight operators

$$\{\gamma^2 C, i\gamma^2 C, \gamma^2 \gamma^4 C, i\gamma^2 \gamma^4 C, \gamma^4, i\gamma^4, i, I\}, \tag{11}$$

where  $C$  is the operator of complex conjugation  $C\psi = \psi^*$  (the operator of involution in  $\mathbb{H}^{3,4}$ ). Operators (11) generate the Pauli-Gürsey-Ibragimov algebra [22,23] over the field of real numbers.

At first we have proved that six of the operators (11)

$$\left\{ s_{\text{PGI}}^{01} = \frac{i}{2} \gamma^2 C, s_{\text{PGI}}^{02} = -\frac{1}{2} \gamma^2 C, s_{\text{PGI}}^{03} = -\frac{i}{2} \gamma^4, s_{\text{PGI}}^{23} = \frac{i}{2} \gamma^2 \gamma^4 C, \right. \\ \left. s_{\text{PGI}}^{31} = -\frac{1}{2} \gamma^2 \gamma^4 C, s_{\text{PGI}}^{12} = -\frac{i}{2} \right\} \tag{12}$$

realize the additional  $D\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$  representation [24–28] of universal covering

$L = \text{SL}(2, \mathbb{C})$  of the proper orthochronous Lorentz group  $L_+^\uparrow = \text{SO}(1,3)$ . Moreover, all operators from the set (10) commute with the operators from (12).

Further, the application of the simplest linear combinations of the generators (10) and (12) enable us [24–28] to find Bose representations of the Lorentz group  $L$  and the Poincaré group  $P \supset L = \text{SL}(2, \mathbb{C})$  (here  $P$  is the universal covering of the proper orthochronous Poincaré group  $P_+^\uparrow \supset L_+^\uparrow = \text{SO}(1,3)$ ), with respect to which the massless Dirac equation is invariant. Thus, we have found the Bose  $D(1,0) \oplus (0,0)$  and  $D(1/2, 1/2)$  representations of the Lie algebra of the Lorentz group  $L$  together with the tensor-scalar of the spin  $s=(1,0)$  and vector representations of the Lie algebra of the Poincaré group  $P$ , with respect to which the Dirac equation with  $m = 0$  is invariant.

After that the goal of further investigations has been formulated as follows. *To find the similar symmetries of the Dirac equation with nonzero mass.* Our first idea in this direction was to find the complete set of combinations of the Pauli-Gürsey-Ibragimov operators (11) and the elements of the  $\gamma$  matrix representation of the  $C\ell^C(1,3)$  algebra demonstrated in the Table 1. This program has been fulfilled in [6–12] and resulted in the 64 elements of the matrix representation of the Clifford algebra  $C\ell^R(0,6)$ . After that among these 64 elements the 28 generators of  $SO(8)$  algebra have been found.

Here let us recall briefly the fact that seven  $\gamma$  matrices

$$\{\gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \gamma^5 = \gamma^1 \gamma^3 C, \gamma^6 = i \gamma^1 \gamma^3 C, \gamma^7 = i \gamma^0\}, \quad (13)$$

satisfy the anti-commutation relations

$$\gamma^A \gamma^B + \gamma^B \gamma^A = -2\delta^{AB}, \quad A, B = \overline{1,7}, \quad (14)$$

of the Clifford algebra generators over the field of real numbers. Due to the evident fact that only six operators of (13) are linearly independent,  $\gamma^4 = -i \gamma^7 \gamma^1 \gamma^2 \gamma^3$ , it is the representation of the Clifford algebra  $C\ell^R(0,6)$  of the dimension  $2^6 = 64$ .

Operators (13) generates also the 28 matrices

$$s^{\overline{AB}} = \left\{ s^{AB} = \frac{1}{4} [\gamma^A, \gamma^B], s^{A8} = -s^{8A} = \frac{1}{2} \gamma^A \right\}, \quad \overline{A}, \overline{B} = \overline{1,8}, \quad (15)$$

which satisfy the commutation relations of the Lie algebra  $SO(8)$

$$[s^{\overline{AB}}, s^{\overline{CD}}] = \delta^{\overline{AC}} s^{\overline{BD}} + \delta^{\overline{CB}} s^{\overline{DA}} + \delta^{\overline{BD}} s^{\overline{AC}} + \delta^{\overline{DA}} s^{\overline{CB}}. \quad (16)$$

It is evident that here we have the algebra over the field of real numbers. Furthermore, it is evident that 28 elements (15) of  $SO(8)$  do not form any Clifford algebra and do not form any subalgebra of the Clifford algebra. It is independent from the Clifford algebra mathematical object.

In our earlier publications on this subject [6-12] the interpretation of the relationship between the  $SO(8)$  algebra and the Clifford algebra  $C\ell^R(0,6)$  was not correct. Nevertheless, the last time publications [17-21] are free already from the interpretation difficulties.

The algebras  $SO(8)$  and  $C\ell^R(0,6)$  are the basis of the proof of the nonzero mass Dirac equation Fermi-Bose duality. Here in (15) we deal with anti-Hermitian operators as well. Nevertheless, the proof started from the Foldy-Wouthuysen representation. Here two sets of operators  $(s^{23}, s^{31}, s^{12})$  and  $(s^{56}, s^{64}, s^{45})$  commute between each other and commute with the operator of the Dirac equation in the Foldy-Wouthuysen representation. Therefore, we can use here the methods developed in [24-28] for the case  $m = 0$ . Note that the subject of [6-12] publications was the Fermi-Bose duality in the sense of [29]. The results [6-12] are not related to the supersymmetry models.

The only problem left to demonstrate similarly the 64 elements of the  $Cl^R(0,6)$  algebra. It is easy to recount them as follows. The first 16 operators are given in the Table 1, the next 16 are found from them with the help of the multiplication by  $i \equiv \sqrt{-1}$ . Last 32 are found from first 32 with the help of multiplication by operator C of complex conjugation.

**Table 2.** 28 generators of the  $\gamma$  matrix representation of the SO(8) algebra.

$\frac{1}{2}\gamma^1\gamma^2$	$\frac{1}{2}\gamma^1\gamma^3$	$\frac{1}{2}\gamma^1\gamma^4$	$\frac{1}{2}\gamma^1\gamma^5$	$\frac{1}{2}\gamma^1\gamma^6$	$\frac{1}{2}\gamma^1\gamma^7$	$\frac{1}{2}\gamma^1\gamma^8 \equiv \frac{1}{2}\gamma^1$
	$\frac{1}{2}\gamma^2\gamma^3$	$\frac{1}{2}\gamma^2\gamma^4$	$\frac{1}{2}\gamma^2\gamma^5$	$\frac{1}{2}\gamma^2\gamma^6$	$\frac{1}{2}\gamma^2\gamma^7$	$\frac{1}{2}\gamma^2\gamma^8 \equiv \frac{1}{2}\gamma^2$
		$\frac{1}{2}\gamma^3\gamma^4$	$\frac{1}{2}\gamma^3\gamma^5$	$\frac{1}{2}\gamma^3\gamma^6$	$\frac{1}{2}\gamma^3\gamma^7$	$\frac{1}{2}\gamma^3\gamma^8 \equiv \frac{1}{2}\gamma^3$
			$\frac{1}{2}\gamma^4\gamma^5$	$\frac{1}{2}\gamma^4\gamma^6$	$\frac{1}{2}\gamma^4\gamma^7$	$\frac{1}{2}\gamma^4\gamma^8 \equiv \frac{1}{2}\gamma^4$
				$\frac{1}{2}\gamma^5\gamma^6$	$\frac{1}{2}\gamma^5\gamma^7$	$\frac{1}{2}\gamma^5\gamma^8 \equiv \frac{1}{2}\gamma^5$
					$\frac{1}{2}\gamma^6\gamma^7$	$\frac{1}{2}\gamma^6\gamma^8 \equiv \frac{1}{2}\gamma^6$
						$\frac{1}{2}\gamma^7\gamma^8 \equiv \frac{1}{2}\gamma^7$

Thus, if to introduce the notation “independent CD” (CD is taken from Clifford-Dirac) for the set of 16 matrices from the Table 1, then the set of 64 elements of  $Cl^R(0,6)$  algebra representation will be given by:

$$\{\text{independent CD} \cup i \cdot \text{independent CD} \cup C \cdot \text{independent CD} \cup iC \cdot \text{independent CD}\}. \quad (17)$$

#### 4. CONCLUSIONS

New extended matrix representations of the Clifford and SO(8) algebras open new possibilities for the development of the quantum field theory.

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