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Analytic spin and pseudospin solutions to the Dirac equation for the Manning-Rosen plus shifted Deng-Fan potential and Yukawa-like tensor interaction

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ABSTRACT

We solve the Dirac equation for the Manning-Rosen plus shifted Deng-Fan potential including a Yukawa-like tensor potential with arbitrary spin–orbit coupling quantum number κ. In the framework of the spin and pseudospin (pspin) symmetry, we obtain the energy eigenvalue equation and the corresponding eigenfunctions in closed form by using the Nikiforov–Uvarov method. Also Special cases of the potential as been considered and their energy eigen values as well as their corresponding eigen functions are obtained for both relativistic and non-relativistic scope.

***Keywords*:** Dirac equation, Manning-Rosen potential, shifted Deng-Fan potential, spin and pseudospin symmetry, Nikiforov-Uvarov Method

1. INTRODUCTION

The Dirac equation (KGE) is the well-known relativistic wave equation that describes spin -½ particles. It is also known that the analytic solutions of the Dirac equation are possible only in a few cases such as harmonic and Coulomb potentials. The presence of a particle in a strong potential field necessitates a relativistic description of such a particle [1,2]. The Dirac equation for the case of exact spin symmetric occurs when the difference in the magnitude between the repulsive vector potential and the attractive scalar potential is zero and the sum of the magnitudes of the repulsive vector potential and attractive scalar potential is equal to a given potential [3].

The exact pseudo-spin symmetry occurs when the sum of the magnitude of the repulsive vector potential and the attractive scalar potential is zero and the difference between the vector potential and scalar potential is equal to a given potential, which is central or non-central. By considering the relativistic case, we can describe the motion of such a particle either by the Klein-Gordon equation or the Dirac equation depending upon the spin character of the particle [4-6]. The Deng-Fan molecular potential is a simple modified Morse potential called the generalized Morse potential, which was proposed by Deng and Fan in 1957 [7] in an attempt to find a more suitable diatomic potential to describe the vibrational spectrum [8]. Although, this potential is qualitatively similar to the Morse potential but it has correct asymptotic behaviour as the internuclear distance approaches to zero [9]. The Deng-Fan potential model can be used to describe the motion of nucleons in the mean field produced by the interactions between nuclei [10]. Recently our group have attempt to study the bound state solutions of Klein-Gordon, Dirac and Schrodinger equations using a combined or mixed potentials. Some of which includes Woods-Saxon plus Attractive Inversely Quadratic potential (WSAIQP) [11], Manning-Rosen plus a class of Yukawa potential (MRCYP) [12], generalied wood-saxon plus Mie-type potential (GWSMP) [13], Kratzer plus Reduced Pseudoharmonic Oscillator potential (KRPHOP) [14].

In this work, our aim is to solve the Dirac equation for the Manning-Rosen plus shifted Deng-Fan (MRsDF) potential in the presence of spin and pspin symmetries and by including a Yukawa-like tensor potential. The MRsDF potential takes the following form:

$$V(r) = -\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] + D_e\left[\frac{b^2}{(e^{\alpha r}-1)^2} - \frac{2b}{(e^{\alpha r}-1)}\right] \quad b = e^{\alpha r_e} - 1, \tag{1a}$$

Thus eq. (1a) can be further expressed as:

$$V(r) = -\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] + D_e\left[\frac{b^2e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{2be^{-\alpha r}}{(1-e^{-\alpha r})}\right] \tag{1b}$$

where: α is the screening parameter, C, D are potential depths, D_e is the Dissociation energy, where: r_e is the equilibrium bond length.

This paper is organized as follows. In section 2, we briefly introduce the Dirac equation with scalar and vector potentials with arbitrary spin–orbit coupling quantum number κ including tensor interaction under spin and pspin symmetry limits. The Nikiforov–Uvarov (NU) method is presented in section 3. The energy eigenvalue equations and corresponding eigenfunctions are obtained in section 4. In this section. In section 5, we discussed some special cases of the potential. Finally, our conclusion is given in section 6.

2. THE DIRAC EQUATION WITH TENSOR COUPLING POTENTIAL

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential $S(r)$, a repulsivevectorpotential $V(r)$ andatensorpotential$U(r)$ (in units ħ = c = 1) is

$$\left[\vec{\alpha}\cdot\vec{p}+\beta\left(M+S(r)\right)-i\beta\vec{\alpha}\cdot\vec{r}U(r)\right]\psi(\vec{r})=[E-V(r)]\psi(\vec{r}). \tag{2}$$

where: E is the relativistic binding energy of the system, $p=-i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given by

$$\vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\ \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \tag{3}$$

were: I is the 2×2 unitary matrix and $\vec{\sigma}$ are three-vector spin matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4}$$

The eigenvalues of the spin–orbit coupling operator are $\kappa=\left(j+\frac{1}{2}\right)>0$ and $\kappa=-\left(j+\frac{1}{2}\right)<0$ for unaligned spin $j=l-\frac{1}{2}$ and aligned spin $j=l+\frac{1}{2}$, respectively. The set (H^2,K,J^2,J_z) can be taken as the complete set of conservative quantities with $\vec{J}$ being the total angular momentum operator and $K=(\vec{\sigma}.\vec{L}+1)$ is the spin–orbit where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian.

Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n,\kappa}(\vec{r})=\begin{pmatrix}f_{n,\kappa}(\vec{r})\\ g_{n,\kappa}(\vec{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{n,\kappa}(r) & Y^{l}_{jm}(\theta,\varphi)\\ iG_{n,\kappa}(r) & Y^{\tilde{l}}_{jm}(\theta,\varphi)\end{pmatrix}, \tag{5}$$

where: $f_{n,\kappa}(\vec{r})$ is the upper (large) component and$g_{n,\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y^{l}_{jm}(\theta,\varphi)$ and $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ are spin and pspin spherical harmonics, respectively, and m is the projection of the angular momentum on the $z-axis$. Substituting equation (5) into equation (2) and making use of the following relations

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right), \tag{6a}$$

$$\left(\vec{\sigma}\cdot\vec{P}\right)=\vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P}+i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{6b}$$

together with the properties

$$\begin{aligned}
&\left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{\hat{l}}(\theta,\varphi)=(\kappa-1)Y_{jm}^{\hat{l}}(\theta,\varphi),\\
&\left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{l}(\theta,\varphi)=-(\kappa-1)Y_{jm}^{l}(\theta,\varphi),\\
&(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\hat{l}}(\theta,\varphi)=-Y_{jm}^{l}(\theta,\varphi),\\
&(\vec{\sigma}\cdot\hat{r})Y_{jm}^{l}(\theta,\varphi)=-Y_{jm}^{\hat{l}}(\theta,\varphi),
\end{aligned} \tag{7}$$

one obtains two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ as:

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n,\kappa}(r)=\left(M+E_{n\kappa}-\Delta(r)\right)G_{n,\kappa}(r), \tag{8a}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r)=\left(M-E_{n\kappa}+\Sigma(r)\right)F_{n,\kappa}(r), \tag{8b}$$

where:

$$\Delta(r)=V(r)-S(r), \tag{9a}$$

$$\Sigma(r)=V(r)+S(r), \tag{9b}$$

After eliminating $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ in equations (8), we obtain the following two Schrodinger-like differential equations for the upper and lower radial spinor components:

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{dU(r)}{dr}-U^2(r)\right]F_{n,\kappa}(r)+\frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n,\kappa}(r)$$
$$=\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)\right]F_{n,\kappa}(r) \tag{10}$$

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{dU(r)}{dr}-U^2(r)\right]G_{n,\kappa}(r)+\frac{\frac{d\Sigma(r)}{dr}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r)=$$
$$\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)\right]G_{n,\kappa}(r), \tag{11}$$

respectively, where $\kappa(\kappa-1)=\hat{l}\left(\hat{l}+1\right)$ and $\kappa(\kappa+1)=l(l+1)$.

The quantum number κ is related to the quantum numbers for spin symmetry l and pspin symmetry $\hat{l}$ as:

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right)\left(s_{1/2}, p_{3/2}, etc\right) \\ j = l + \frac{1}{2},\ \ aligned\ spin\ (\kappa < 0), \\ +l = +\left(j+\frac{1}{2}\right)\left(p_{1/2}, d_{3/2}, etc\right) \\ j = l - \frac{1}{2},\ \ unaligned\ spin\ (\kappa > 0)\ , \end{cases} \tag{12}$$

and the quasidegenerate doublet structure can be expressed in terms of a pspin angular momentum $\hat{s} = 1/2$ and pseudo-orbital angular momentum $\hat{l}$, which is defined as

$$\kappa = \begin{cases} -\hat{l} = -\left(j+\frac{1}{2}\right)\left(s_{1/2}, p_{3/2}, etc\right) \\ j = \hat{l} - \frac{1}{2},\ \ aligned\ spin\ (\kappa < 0), \\ +(\hat{l}+1) = +\left(j+\frac{1}{2}\right)\left(d_{3/2}, f_{5/2}, etc\right) \\ j = \hat{l} + \frac{1}{2},\ \ unaligned\ spin\ (\kappa > 0)\ , \end{cases} \tag{13}$$

where: $\kappa = \pm 1, \pm 2, \ldots$. For example, $(1s_{1/2}, 0d_{3/2})$ and $(0p_{3/2}, 0f_{5/2})$ can be considered as pspin doublets

2. 1. Spin symmetry limit

In the spin symmetry limit, $\frac{d\Delta(r)}{dr} = 0$ or $\Delta(r) = C_s$ =constant, with $\Sigma(r)$ taking as the MRsDF potential eq. (1b) and the Yukawa-like tensor potential. i.e

$$\Sigma(r) = V(r) = -\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] + D_e\left[\frac{b^2 e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{2be^{-\alpha r}}{(1-e^{-\alpha r})}\right] \tag{14}$$

$$U(r) = -\frac{H}{r}e^{-\alpha r}, \tag{15}$$

Under this symmetry, equation (10) is recast in the simple form

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa He^{-\alpha r}}{r^2} - \frac{He^{-\alpha r}}{r^2} - \frac{\alpha He^{-\alpha r}}{r} - \frac{H^2 e^{-2\alpha r}}{r^2}\right] F_{n,\kappa}(r) = \left[\gamma\left(-\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] + D_e\left[\frac{b^2 e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{2be^{-\alpha r}}{(1-e^{-\alpha r})}\right]\right) + \beta^2\right] F_{n,\kappa}(r)$$

(16a)

where $\kappa = l$ and $\kappa = -l - 1$ for κ < 0 and κ > 0, respectively. Also, $\gamma = (M + E_{n\kappa} - C_s)$ and $\beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_s)$. (16b)

2. 2. Pseudospin symmetry limit

Ginocchio[] showed that there is a connection between pspin symmetry and near equality of the time component of a vector potential and the scalar potential, $V(r) \approx -S(r)$.

After that, Meng et al [,] derived that if $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps}$ =constant, then pspin symmetry is exact in the Dirac equation. Here, we are taking $\Delta(r)$ as the MRE potential eq. (1) and the tensor potential as the Yukawa-like potential. Thus, equation (11) is recast in the simple form

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H e^{-\alpha r}}{r^2} + \frac{He^{-\alpha r}}{r^2} + \frac{\alpha He^{-\alpha r}}{r} - \frac{H^2 e^{-2\alpha r}}{r^2}\right] G_{n,\kappa}(r) = \left[\gamma\left(-\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] + D_e\left[\frac{b^2 e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{2be^{-\alpha r}}{(1-e^{-\alpha r})}\right]\right) + \beta^2\right] G_{n,\kappa}(r) \tag{17a}$$

where: $\kappa = -\tilde{l}$ and $\kappa = \tilde{l} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Also, $\tilde{\gamma} = \left(E_{n\kappa} - M - C_{ps}\right)$ and $\tilde{\beta}^2 = (M + E_{n\kappa})\left(M - E_{n\kappa} + C_{ps}\right)$. (17b)

to obtain the analytic solution, we use an approximation for the centrifugal term as:

$$\frac{1}{r^2} = \frac{\alpha^2}{(1-e^{-\alpha r})^2} \tag{18}$$

Finally, for the solutions to equations (16) and (17) with the above approximation, we will employ the NU method, which is briefly introduced in the following section

3. THE NIKIFOROV–UVAROV METHOD

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi_n'(s) + \frac{\overline{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \tag{19}$$

where: $\sigma(s)$ and $\overline{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3\right]\Psi(s) = 0 \tag{20}$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1-c_3 s), \overline{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \tag{21}$$

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \tag{22}$$

$$(c_2 - c_3)n + c_3n^2 - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \tag{23}$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l}S^{c_{12}}(1-c_3s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3s) \tag{24}$$

where:

$$c_4 = \frac{1}{2}(1-c_1),\ c_5 = \frac{1}{2}(c_2 - 2c_3),\ c_6 = c_5{}^2 + \epsilon_1,\ c_7 = 2c_4c_5 - \epsilon_2,\ c_8 = c_4{}^2 + \epsilon_3,$$
$$c_9 = c_3c_7 + c_3{}^2c_8 + c_6,\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c3\sqrt{c_8}\right)$$
$$c_{12} = c_4 + \sqrt{c_8},\ c_{13} = c_5 - \left(\sqrt{c_9} + c3\sqrt{c_8}\right) \tag{25}$$

and P_n is the orthogonal polynomials.

4. SOLUTIONS TO THE DIRAC EQUATION

We will now solve the Dirac equation with the MRsDF potential and tensor potential by using the NU method.

4. 1. The spin symmetric case

To obtain the solution to equation (16), by using the transformation $s = e^{-\alpha r}$, we rewrite it as follows:

$$\frac{d^2F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\Big[-\kappa(\kappa+1) - 2\kappa Hs - 2Hs + Hs^2 - H^2s^2 + \frac{\gamma}{\alpha^2}\left(Cs + Ds^2 - D_eb^2s^2 + 2D_ebs(1-s)\right) - \frac{\beta^2}{\alpha^2}(1-s)^2\Big]F_{n,\kappa}(s) = 0, \tag{26}$$

Eq. (26) is further simplified as

$$\frac{d^2F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\left(\frac{\beta^2}{\alpha^2} - \frac{2\gamma D_eb}{\alpha^2} + \frac{\gamma D_eb^2}{\alpha^2} - \frac{\gamma D}{\alpha^2} + H^2 - H\right)s^2 + \left(\frac{2\beta^2}{\alpha^2} + \frac{2\gamma D_eb}{\alpha^2} + \frac{\gamma C}{\alpha^2} - 2\kappa H - 2H\right)s - \left(\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)\right)\right]F_{n,\kappa}(s) = 0, \tag{27}$$

Comparing eq. (27) with eq. (20), we obtain

$$c_1 = 1,\quad \epsilon_1 = \frac{\beta^2}{\alpha^2} - \frac{2\gamma D_eb}{\alpha^2} + \frac{\gamma D_eb^2}{\alpha^2} - \frac{\gamma D}{\alpha^2} + H^2 - H$$

$$c_2 = 1, \quad \epsilon_2 = \frac{2\beta^2}{\alpha^2} + \frac{2\gamma D_e b}{\alpha^2} + \frac{\gamma C}{\alpha^2} - 2\kappa H - 2H \tag{28}$$
$$c_3 = 1, \quad \epsilon_3 = \frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)$$

and from eq. (25), we further obtain

$$c_4 = 0, \qquad c_5 = -\frac{1}{2},$$
$$c_6 = \frac{1}{4} + \frac{\beta^2}{\alpha^2} - \frac{2\gamma D_e b}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2} - \frac{\gamma D}{\alpha^2} + H^2 - H, \qquad c_7 = -\left(\frac{2\beta^2}{\alpha^2} + \frac{2\gamma D_e b}{\alpha^2} + \frac{\gamma C}{\alpha^2} - 2\kappa H - 2H\right),$$
$$c_8 = \frac{\beta^2}{\alpha^2} + \kappa(\kappa+1), \qquad c_9 = \left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma D}{\alpha^2} - \frac{\gamma C}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2}, \text{ where } \eta_\kappa = \kappa + H + 1,$$
$$c_{10} = 1 + 2\sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)},$$
$$c_{11} = 2 + 2\left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma D}{\alpha^2} - \frac{\gamma C}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2}} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right), \tag{29}$$
$$c_{12} = \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)},$$
$$c_{13} = -\frac{1}{2} - \left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma D}{\alpha^2} - \frac{\gamma C}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2}} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right)$$

In addition, the energy eigenvalue equation can be obtained by using eq. (23) as follows:

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma D}{\alpha^2} - \frac{\gamma C}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2}} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right)^2 = \frac{\beta^2}{\alpha^2} - \frac{\gamma D}{\alpha^2} + \frac{2\gamma D_e b}{\alpha^2} + \frac{\gamma D_e b^2}{\alpha^2} + H^2 - H \tag{30}$$

By substituting the explicit forms of $\gamma\ and\ \beta^2$ after equation (16) into equation (30), one can readily obtain the closed form for the energy formula.

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2}(M + E_{n\kappa} - C_s) - \frac{C}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{D_e b^2}{\alpha^2}(M + E_{n\kappa} - C_s)} + \sqrt{\frac{1}{\alpha^2}\big((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\big) + \kappa(\kappa+1)}\right)^2 = \frac{1}{\alpha^2}\big((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\big) - \frac{D}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{D_e b^2}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{2D_e b}{\alpha^2}(M + E_{n\kappa} - C_s) + H^2 - H \tag{31}$$

On the other hand, to find the corresponding wave functions, referring to equation (29) and eq. (24), we obtain the upper component of the Dirac spinor from eq. 24 as”

$$F_{n,\kappa}(s) = B_{n,\kappa} s^{\sqrt{\frac{\beta^2}{\alpha^2}+\kappa(\kappa+1)}} (1-s)^{\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\frac{\gamma D}{\alpha^2}-\frac{\gamma C}{\alpha^2}+\frac{\gamma D_e b^2}{\alpha^2}}} P_n^{\left(2\sqrt{\frac{\beta^2}{\alpha^2}+\kappa(\kappa+1)},2\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\frac{\gamma D}{\alpha^2}-\frac{\gamma C}{\alpha^2}+\frac{\gamma D_e b^2}{\alpha^2}}\right)}(1-2s) \quad (32)$$

where $B_{n,\kappa}$ is the normalization constant. The lower component of the Dirac spinor can be calculated from equation (8a)

$$G_{n,\kappa}(r) = \frac{1}{(M+E_{n\kappa}-C_s)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n,\kappa}(r) \quad (33)$$

where: $E_{n\kappa} \neq -M + C_s$.

4. 2. The pseudospin symmetric case

To avoid repetition in the solution of equation (17), we follow the same procedures explained in section 4.1and hence obtain the following energy eigenvalue equation:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{\tilde{\gamma} D}{\alpha^2}-\frac{\tilde{\gamma} C}{\alpha^2}+\frac{\tilde{\gamma} D_e b^2}{\alpha^2}}+\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\kappa(\kappa-1)}\right)^2 = \frac{\tilde{\beta}^2}{\alpha^2}-\frac{\tilde{\gamma} D}{\alpha^2}+\frac{\tilde{\gamma} D_e b^2}{\alpha^2}+\frac{2\tilde{\gamma} D_e b}{\alpha^2}+H^2+H \quad (34)$$

By substituting the explicit forms of $\tilde{\gamma}$ *and* $\tilde{\beta}^2$after equation (17b) into equation (34), one can readily obtain the closed form for the energy formula as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{D}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)-\frac{C}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{D_e b^2}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)}+\sqrt{\frac{1}{\alpha^2}\left(\left(M+E_{n\kappa}\right)\left(M-E_{n\kappa}+C_{ps}\right)\right)+\kappa(\kappa-1)}\right)^2 = \frac{1}{\alpha^2}\left(\left(M+E_{n\kappa}\right)\left(M-E_{n\kappa}+C_s\right)\right)-\frac{D}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{D_e b^2}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{2D_e b}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+H^2+H \quad (35)$$

and the corresponding wave functions for the upper Dirac spinor as

$$G_{n,\kappa}(r) = \tilde{B}_{n,\kappa} s^{\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\kappa(\kappa-1)}} (1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{\tilde{\gamma} D}{\alpha^2}-\frac{\tilde{\gamma} C}{\alpha^2}+\frac{\tilde{\gamma} D_e b^2}{\alpha^2}}} P_n^{\left(2\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\kappa(\kappa-1)},2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{\tilde{\gamma} D}{\alpha^2}-\frac{\tilde{\gamma} C}{\alpha^2}+\frac{\tilde{\gamma} D_e b^2}{\alpha^2}}\right)}(1-2s) \quad (36)$$

were $\Lambda_\kappa = \kappa + H$ and $\tilde{B}_{n,\kappa}$ is the normalization constant. Finally, the Upper-spinor component of the Dirac equation can be obtained via equation (8b) as

$$F_{n,\kappa}(r) = \frac{1}{(M-E_{n\kappa}+C_{ps})}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r) \quad (37)$$

where: $E_{n\kappa} \neq M + C_{ps}$

5. DISCUSSIONS

In this section, we are going to study some special cases of the energy eigenvalues given by Eqs. (31) and (35) for the spin and pseudospin symmetries, respectively.

Case 1. If one sets $C_s = 0\,, C_{ps} = 0,\ D_e = 0$ in eq. (31) and eq. (35), we obtain the energy equation of Manning-Rosen potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\frac{D}{\alpha^2}(M+E_{n\kappa})-\frac{C}{\alpha^2}(M+E_{n\kappa})+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)-\frac{D}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (38)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{D}{\alpha^2}(E_{n\kappa}-M)-\frac{C}{\alpha^2}(E_{n\kappa}-M)+\sqrt{\frac{1}{\alpha^2}(M+E_{n\kappa})(M-E_{n\kappa})+\kappa(\kappa-1)}}\right)^2=\frac{1}{\alpha^2}(M+E_{n\kappa})(M-E_{n\kappa})-\frac{D}{\alpha^2}(E_{n\kappa}-M)+H^2+H \quad (39)$$

Case 2. If one sets $C_s = 0\,, C_{ps} = 0,\ C = D = 0$ in eq. (31) and eq. (35), we obtain the energy equation of shifted Deng-Fan potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{D_e b^2}{\alpha^2}(M+E_{n\kappa}-C_s)+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa}-C_s)\right)+\frac{D_e b^2}{\alpha^2}(M+E_{n\kappa}-C_s)+\frac{2D_e b}{\alpha^2}(M+E_{n\kappa}-C_s)+H^2-H \quad (40)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D_e b^2}{\alpha^2}(E_{n\kappa}-M-C_{ps})+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\kappa(\kappa-1)}}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa}+C_s)\right)-$$

$$\frac{D}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{D_e b^2}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{2D_e b}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+H^2+H \tag{41}$$

Case 3. If one sets $C_s = 0$, $C_{ps} = 0$, $B = 0$, $C = 0$, $D = 0$, in eq. (31) and eq. (35), we obtain the energy equation of Hulthen potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n+\eta_\kappa+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{A}{\alpha^2}(M+E_{n\kappa})+H^2-H \tag{42}$$

and

$$\left(n+\Lambda_\kappa+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)\frac{A}{\alpha^2}(M+E_{n\kappa})+H^2+H \tag{43}$$

Case 4. Let us now discuss the relativistic limit of the energy eigenvalues and wavefunctions of our solutions. If we take $C_s = 0, H = 0, \kappa \to l$ and put $S(r) = V(r) = \Sigma(r)$, the nonrelativistic limit of energy equation 31 for MRE potential and wave function 32 under the following appropriate transformations $M + E_{n\kappa} \to \frac{2\mu}{\hbar^2}$, and $M - E_{n\kappa} \to -E_{nl}$ becomes

$$E_{nl}=-\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu A}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2-l(l+1)\right\}$$

and the associated wave functions $F_{n\kappa}(s) \to R_{n,l}(s)$ are

$$R_{n,l}(s)=N_{n,l}s^{U/2}(1-s)^{(V-1)/2}P_n^{(U,V)}(1-2s), \tag{44}$$

where: $U = 2\sqrt{\frac{2\mu E_{nl}}{\alpha^2\hbar^2}+l(l+1)}$ and $V = 2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}}$ (45)

Case 5. If $A = B = 0$ in eq. (44), we obtain the energy equation of Manning-Rosen potential in the non-relativistic limit

$$E_{nl}=-\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu C}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}}}\right]^2-l(l+1)\right\} \tag{46}$$

Case 6. If $C = D = 0$ in eq. (44), we obtain the energy equation of the Eckart potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2 - l(l+1)\right\} \tag{47}$$

Case 7. If $B = 0, C = D = 0$ in eq. (47), we obtain the energy equation of the Hulthen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2}}\right]^2 - l(l+1)\right\} \tag{48}$$

6. CONCLUSION

In this research article, we have studied Analytic spin and pseudospin solutions to the Dirac equation for the Manning-Rosen plus shifted Deng-Fan potential and Yukawa-like tensor interaction. We have obtained the energy eigenvalue equations and the related two-component spinor wave functions with the help of Nikiforov–Uvarov method.

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