



---

---

SHORT COMMUNICATION

## **Bound State Solutions of the s-wave Schrodinger Equation for Generalized Woods-Saxon plus Mie-Type Nuclei Potential within the framework of Nikiforov-Uvarov Method**

**Benedict Iserom Ita<sup>1</sup> and Louis Hitler<sup>1,2,\*</sup>**

<sup>1</sup>Physical/Theoretical Chemistry Research Group, Department of Pure and Applied Chemistry, University of Calabar, P.M.B 1115 Calabar, Cross River State, Nigeria

<sup>1,2</sup>National Centre For Nanoscience and Technology, University of Chinese Academy of Sciences UCAS, Beijing, China

\*E-mail address: [louis@nanoctr.cn](mailto:louis@nanoctr.cn)

### **ABSTRACT**

The solutions of the Schrödinger equation with Generalized Woods-Saxon plus Mie-type potentials (GWSMP) have been presented using the Pekeris-like approximation of the coulomb term and parametric Nikiforov-Uvarov (NU) method. The bound state energy eigenvalues and the corresponding un-normalized eigen functions are obtained in terms of Jacobi polynomials. Special cases of potential consideration have also been discussed.

**Keywords:** Woods-Saxon, Mie-type, Schrodinger, Bound state, Eigen functions, Nikiforov-Uvarov

### **1. INTRODUCTION**

It is well known that the exact solutions play an important role in quantum mechanics since they contain all the necessary information regarding the quantum model under study.

The exact solutions of the Schrodinger equation are only possible for a hydrogen atom and for a harmonic oscillator in three dimensions (Dong, 2007). However, when a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics (Zhang, 2005). Taking the relativistic effects into account, a particle in a potential field should be described with the Klein-Gordon (KG) and Dirac equations. In recent years, there has been an increased interest in finding exact solutions to Schrodinger, KG, Dirac and Salpeter equations for various potential schemes (Ihkdaïr, 2009; Louis *et al.*, 2016). The Woods-Saxon (WS) potential and its various modifications have played an essential role in microscopic physics in the determination of the energy level spacing, particle number dependence of energy quantities and universal properties electron distributions in atoms, nuclei and atomic clusters since it can be used to describe the interaction of neutron with one heavy-ion nucleus and also for optical potential model (Sever, 2005; Ita *et al.*, 2017).

Furthermore, within the framework of the spin and pseudospin symmetry, the analytical solutions of the Woods-Saxon and its generalized versions are 2 solved for Dirac equation and obtained the bound state energies with their corresponding wave functions for the particle and antiparticle (Candemir, 2014). The generalized symmetric Woods-Saxon (GSWS) potential model is more flexible and useful model than the Woods-Saxon potential in order to examine the scattering, bound and quasi-bound state solutions of the wave equations so that the GSWS potential can be applied to physical phenomena such as the scattering, transmission resonance, supercriticality, decay, fusion, fission etc. (Bayrak *et al.*, 2015). GSWS potential in one dimension in case of the scattering, bound and quasi-bound states. Moreover, we examine correlations between the potential parameters with the reflection-transmission coefficients in the case of scattering state, with the energy eigenvalues and their corresponding wave functions in the case of bound state, and finally with the resonance energy eigenvalues and their corresponding wave functions in the case of quasi-bound states (Bayrak *et al.*, 2015). Recently our group have attempt to study the bound state solutions of Klein-Gordon, Dirac and Schrodinger equations using a combined or mixed potentials. Some of which includes Woods-Saxon plus Attractive Inversely Quadratic potential (WSAIQP) (Ita *et al.*, 2017), Manning-Rosen plus a class of Yukawa potential (MRCYP) (Louis *et al.*, 2017), generalized wood-saxon plus Mie-type potential (GWSMP) (Ita *et al.*, 2017), Kratzer plus Reduced Pseudoharmonic Oscillator potential (KRPHOP) (Louis *et al.*, 2017). In this study, we are going to continue with our previous work on approximate s-wave solutions of the Schrodinger equation for Generalized Woods-Saxon plus Mie-type molecular potentials (GWSMP) using the Nikiforov-Uvarov method.

## **2. REVIEW OF PARAMETRIC NIKIFAROV-UVAROV METHOD**

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The Nikiforov-Uvarov method has been successfully applied to relativistic and nonrelativistic quantum mechanical problems and other field of studies as well [6-9].The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi_n'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)} \Psi_n(s) = 0 \quad (1)$$

where  $\sigma(s)$  and  $\bar{\sigma}(s)$  are polynomials at most second degree and  $\tilde{\tau}(s)$  is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi'''(s) + \frac{c_1 - c_2s}{s(1 - c_3s)} \Psi''(s) + \frac{1}{s^2(1 - c_3s)^2} [-\epsilon_1s^2 + \epsilon_2s - \epsilon_3] \Psi(s) = 0 \tag{2}$$

Thus eqn. (1) can be solved by comparing it with equation (2) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2s), \sigma(s) = s(1 - c_3s), \bar{\sigma}(s) = -\epsilon_1s^2 + \epsilon_2s - \epsilon_3 \tag{3}$$

The parameters obtainable from equation (3) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \tag{4}$$

$$(c_2 - c_3)n + c_3n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \tag{5}$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l} S^{c_{12}} (1 - c_3s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10}-1)} (1 - 2c_3s) \tag{6}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5^2 + \epsilon_1, c_7 = 2c_4c_5 - \epsilon_2, c_8 = c_4^2 + \epsilon_3,$$

$$c_9 = c_3c_7 + c_3^2c_8 + c_6, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8})$$

$$c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \tag{7}$$

and  $P_n$  is the orthogonal polynomials.

$$\text{Given that } P_n^{(\alpha,\beta)} = \sum_{r=0}^n \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\beta-r+1)(n-r)!r!} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \tag{8}$$

This can also be expressed in terms of the Rodriguez's formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!} (x-1)^{-\alpha} (x+1)^{-\beta} \left(\frac{d}{dx}\right)^n ((x-1)^{n+\alpha} (x+1)^{n+\beta})$$

### 3. SOLUTION OF S-WAVE SCHRODINGER EQUATION WITH GENERALIZED WOODS-SAXON PLUS MIE-TYPE POTENTIAL (GWSMP)

The s-wave Schrodinger Equation with vector  $V(r)$ , potential is given as

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} [(E - V(r))]R(r) = 0 \tag{9}$$

The generalized Woods-Saxon Potential is given as

$$V(r) = \left( \frac{V_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})} - \frac{V_2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} \right) \tag{10}$$

Similarly, the Mie-type Potential is given as,

$$V(r) = -\frac{A}{r} + \frac{B}{r^2} + C \tag{11}$$

Applying the transformation  $S = e^{-2\alpha r}$ , the superposed potential can be represented as

$$V(s) = \frac{V_1 s}{(1-s)} - \frac{V_1 s^2}{(1-s)^2} - \frac{2A\alpha}{(1-s)} + \frac{4B\alpha^2 s}{(1-s)^2} + C \tag{12}$$

Substitute eq. (12) into eq. (9), we have

$$\frac{d^2 R(s)}{ds^2} + \frac{2\mu}{\hbar^2} \left[ \left( E - \left( \frac{V_1 s}{(1-s)} - \frac{V_1 s^2}{(1-s)^2} - \frac{2A\alpha}{(1-s)} + \frac{4B\alpha^2 s}{(1-s)^2} + C \right) (s) \right) \right] R(s) = 0 \tag{13}$$

Applying the pekeris-type approximation and after lengthy algebra, we have

$$\frac{d^2 R(s)}{ds^2} + \frac{(1-s)}{(1-s)s} \frac{dR(s)}{ds} + \frac{1}{(1-s)^2 s^2} [(2\beta^2 - B + P - K)s^2 + (-H - B + \phi - 4\beta^2)s + (2\beta^2 - 2J - K + H)] R(s) = 0 \tag{14}$$

where

$$-\beta^2 = \left( \frac{\mu E}{4\alpha^2 \hbar^2} \right), B = \left( \frac{\mu}{2\alpha^2 \hbar^2} \right) V_1, P = \left( \frac{\mu}{2\alpha^2 \hbar^2} \right) V_2, \phi = \left( \frac{\mu}{\alpha^2 \hbar^2} \right) C, H = \left( \frac{\mu}{\alpha \hbar^2} \right) A, J = \left( \frac{\mu}{\hbar^2} \right) B, K = \left( \frac{\mu}{2\alpha^2 \hbar^2} \right) C \tag{15}$$

$$\begin{aligned} c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + 2\beta^2 - B - P + K, c_7 = -4\beta^2 + B + H - \phi, \\ c_8 = 2\beta^2 - 2J - K + H, c_9 = \frac{1}{4} - \phi - 2J - P, c_{10} = 1 + 2\sqrt{2\beta^2 - 2J - K + H}, \\ c_{11} = 2 + 2 \left( \sqrt{\frac{1}{4} - \phi - 2J - P} + \sqrt{2\beta^2 - 2J - K + H} \right), \\ c_{12} = \sqrt{2\beta^2 - 2J - K + H}, c_{13} = -\frac{1}{2} - \left( \sqrt{\frac{1}{4} - \phi - 2J - P} + \sqrt{2\beta^2 - 2J - K + H} \right), \\ \varepsilon_1 = 2\beta^2 + B + P + K, \varepsilon_2 = 4\beta^2 - \phi + B + H, \\ \varepsilon_3 = 2\beta^2 - 2J - K + H \end{aligned} \tag{16}$$

Using the eigenvalue equation, the energy eigen spectrum of GWSMTP is found to be

$$\beta^2 = \left[ \frac{(\phi + 2K + 4J) - (B + 3H) - (n^2 + n - \frac{1}{2}) - (2n + 1) \sqrt{\frac{1}{4} - \phi - 2J - P}}{(n + \frac{1}{2}) + 2\sqrt{\frac{1}{4} - \phi - 2J - P}} \right]^2 - (2J + K - H) \tag{17}$$

The above equation can be solved explicitly and the energy eigen spectrum of GWSMTP becomes

$$E = \frac{2\alpha^2\hbar^2}{\mu} \left\{ \frac{\left[ \left( \left( \frac{\mu}{\alpha^2\hbar^2} \right) C + \left( \frac{\mu}{\alpha^2\hbar^2} \right) C + 4 \left( \frac{\mu}{\hbar^2} \right) B \right) - \left( \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_1 + 3 \left( \frac{\mu}{\alpha\hbar^2} \right) A \right) - (n^2 + n + \frac{1}{2}) - (2n+1) \sqrt{\frac{q^2}{4} - \left( \frac{\mu}{\alpha^2\hbar^2} \right) C - 2 \left( \frac{\mu}{\hbar^2} \right) B - \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_2} \right]}{(n + \frac{1}{2}) + 2 \sqrt{\frac{1}{4} - \left( \frac{\mu}{\alpha^2\hbar^2} \right) C - 2 \left( \frac{\mu}{\hbar^2} \right) B - \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_2}} \right\} - \left( 2 \left( \frac{\mu}{\hbar^2} \right) B + \left( \frac{\mu}{2\alpha^2\hbar^2} \right) C - \left( \frac{\mu}{\alpha\hbar^2} \right) A \right) \quad (18)$$

Eq. (18) is the bound state eigen energy of the superposed potentials under investigation.

**Eigen function consideration**

$$\rho(s) = s^u(1 - qs)^v \quad (19)$$

where  $u = 2\beta^2 - R - B + \lambda$ , and  $v = 2\sqrt{\frac{1}{4} - B - H - P + \lambda}$

$$X_n(s) = p_n^{(u,v)}(1 - 2s), \quad (20)$$

$$\varphi(s) = s^{u/2}(1 - s)^{1+v/2} \quad (21)$$

Using equation (16) we get the function  $\chi(s)$  as

$$\chi(s) = P_n^{(U,V)}(1 - 2s), \quad (22)$$

where  $P_n^{(U,V)}$  are Jacobi polynomials Lastly,

$$\varphi(s) = s^{c_{12}}(1 - c_3s)^{-c_{12} - \frac{c_{13}}{c_3}}, \quad (23)$$

and using equation (16) we get

$$\varphi(s) = s^{U/2}(1 - s)^{V-1/2}, \quad (24)$$

We then obtain the radial wave function from the equation

$$R_n(s) = N_n\varphi(s)\chi_n(s),$$

As

$$R_n(s) = N_n s^{U/2}(1 - s)^{(V-1)/2} P_n^{(U,V)}(1 - 2s), \quad (25)$$

where  $n$  is a positive integer and  $N_n$  is the normalization constant.

#### 4. DISCUSSION

##### Special Cases of Potential Considerations:

**CASE I: when  $V_1 = V_2 = 0$**  eq. (18) is reduced to bound state equation with a pure Mie-type potential given as follows

$$E = \frac{2\alpha^2\hbar^2}{\mu} \left\{ \left[ \frac{\left( \left( \frac{\mu}{\alpha^2\hbar^2} \right) C + \left( \frac{\mu}{\alpha^2\hbar^2} \right) C + 4 \left( \frac{\mu}{\hbar^2} \right) B \right) - \left( 3 \left( \frac{\mu}{\alpha\hbar^2} \right) A \right) - \left( n^2 + n + \frac{1}{2} \right) - (2n+1) \sqrt{\frac{q^2}{4} - \left( \frac{\mu}{\alpha^2\hbar^2} \right) C - 2 \left( \frac{\mu}{\hbar^2} \right) B}}{\left( n + \frac{1}{2} \right) + 2 \sqrt{\frac{1}{4} - \left( \frac{\mu}{\alpha^2\hbar^2} \right) C - 2 \left( \frac{\mu}{\hbar^2} \right) B}} \right] - \left( 2 \left( \frac{\mu}{\hbar^2} \right) B + \left( \frac{\mu}{2\alpha^2\hbar^2} \right) C - \left( \frac{\mu}{\alpha\hbar^2} \right) A \right) \right\} \quad (26)$$

**CASE II: when  $A = B = C = 0$**  eq. (18) is reduced to bound state solution with a pure Generalized Woods-saxon equation given as

$$E = \frac{2\alpha^2\hbar^2}{\mu} \left\{ \left[ \frac{\left( \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_1 \right) - \left( n^2 + n + \frac{1}{2} \right) - (2n+1) \sqrt{\frac{1}{4} - \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_2}}{\left( n + \frac{1}{2} \right) + 2 \sqrt{\frac{1}{4} - \left( \frac{\mu}{2\alpha^2\hbar^2} \right) V_2}} \right] \right\} \quad (27)$$

#### 5. CONCLUSION

In this paper, we investigate Approximate the bound state solutions of Schrodinger equation with Mie-type plus Generalized woods-saxon potentials be applying the pekeris-like approximation scheme. We obtained energy eigenvalues and the corresponding unnormalized wave function in terms of Jacobi polynomials. Special cases for the potentials are discussed indicating usefulness for other physical systems.

#### References

- [1] S. H. Dong, Factorization Method in Quantum Mechanics, Springer, 2007
- [2] X.-C. Zhang, Q.-W. Liu, C.-S. Jia and L.-Z. Wang. *Phys. Lett. A* 340, 59 (2005)
- [3] S. M. Ikhdair and R. Sever. *Phys. Scr.* 79 (3), 035002 (2009)
- [4] C. Berkdemir, A. Berkdemir and R. Sever. *Phys. Rev. C* 72, 027001 (2005)

- [5] Candemir, N., Bayrak, O., Bound states of the Dirac equation for the generalized Woods-Saxon potential in pseudospin and spin symmetry limits. *Mod. Phys. Lett. A*29, 1450180 (2014).
- [6] B. C. Lütüoğlu, F. Akdeniz and O. Bayrak. *Phys. Scr.* 90, 015302 (2015)
- [7] B. I. Ita, H. Louis, T. O. Magu and N. A. Nzeata-Ibe. Bound state solutions of Klein-Gordon equation with Woods-Saxon plus Attractive Inversely Quadratic potential Via parametric Nikifarov-Uvarov method. *World Scientific News* 74 (2017) 280-287
- [8] H. Louis, B.I. Ita, T. O. Magu and N. A Nzeata-Ibe. Bound state solutions of the Schrodinger equation with Manning-Rosen plus a class of Yukawa potential using Pekeris-like approximation of the coulombic term and parametric Nikifarov-Uvarov method. *World Scientific News* 70(2) (2017) 312-319
- [9] B. I. Ita, B. E. Nyong., H. Louis, T. O. Magu (2017). Radial solution of the s-wave Klein-Gordon equation for generalied wood-saxon plus Mie-type potential using Nikifarov-Uvarov. *J. Chem. Soc. Nigeria*, Vol. 41, No. 2, pp. 21-26
- [10] H. Louis, B. I. Ita., B.E. Nyong, T. O. Magu, S. Barka and N.A. Nzeata-Ibe. Radial solution of the s-wave D-dimensional Non-Relativistic Schrodinger equation for generalized manning-Rosen plus Mie-type nuclei potentials within the framewoark of parametric Nikifarov-Uvarov Method. *Journal of Nigerian Association of Mathematical Physics* Vol. 36, No. 2, (2016) 193-198
- [11] B. I. Ita, B.E. Nyong, N. O. Alobi, H. Louis and T. O. Magu. Bound State Solution of the Klein-Gordon Equation for Modified Echart Plus Inverse Square Molecular Potential with Improved new Approximation Scheme to Centrifugal Term. *Equatorial Journal of Computational and Theoretical Sciences*, Volume 1, Issue 1, (2016) 55-64.
- [12] B. I. Ita, A.I. Ikeuba, H. Louis and P. Tchoua. Solution of the Schrodinger equation with inversely quadratic Yukawa plus attractive radial potential using Nikifarov-Uvarov method. *Journal of Theoretical Physics and Cryptography*. Vol. 10, December, 2015

( Received 14 June 2017; accepted 03 July 2017 )