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SHORT COMMUNICATION

Bound State Solutions of the Klein Gordon Equation with Woods-Saxon Plus Attractive Inversely Quadratic Potential Via Parametric Nikiforov-Uvarov Method

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ABSTRACT

We study the bound state solutions of the Klein-Gordon equation with Woods-Saxon plus attractive inversely quadratic potential using the parametric Nikiforov-Uvarov Method. We obtained the bound state energy eigenvalues and the corresponding normalized eigen functions expressed in terms of hypergeometric functions. Two special cases of this potential are discussed.

***Keywords*:** Bound state solutions, Klein-gordon equation, Nikiforov-Uvarov method, Woods-saxon potential, Attractive inversely quadratic potential

1. INTRODUCTION

Over the years theoretical physics and Chemistry have been successful in explaining the behaviour of different particles in different potentials. This has been made possible through obtaining exact or approximate solutions of the non-relativistic and relativistic wave equations for different physical systems of interest (Louis *et al*., 2016). In nuclear and high energy physics, one of the interesting problems is to obtain exact solution of the Klein - Gordon, Duffin – Kemmer - Petiau and Dirac equations for mixed vector and scalar potentials. When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non – relativistic quantum mechanics (Ita *et al*., 2017). The Klein – Gordon, Dirac, and Duffin – Kemmer – Petiau wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics (Bayrak and Sahin, 2015). In relativistic quantum mechanics, one can apply the Klein - Gordon equation to the treatment of a zero-spin particle and the Dirac equation for spin half particle. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and corresponding wave functions of the Klein-Gordon and Dirac equations (Louis *et al.*, 2017).

These relativistic equations contain two objects: the four – vector linear momentum operation and the scalar rest mass. These allow one to introduce two types of potential coupling, which are the four vector potential V(r) and the space – time scalar potential V(r). The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$\left[-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+M)^2\right]\psi(r,\theta,\phi)=0$$

where M is the rest mass, $i\frac{\partial}{\partial t}$ = energy eigen value, $V(r)$ and $S(r)$ are the vector and scalar potentials respectively.

However, the analytical solutions of the Klein-Gordon equation are possible only in the s-wave case with the angular momentum l = 0 for some well known potential. Conversely, when $l \neq 0$, one can only solve approximately the Klein-Gordon equation for some potentials using a suitable approximation scheme (mohammet and suleyman, 2016)

Different methods have been employed to obtain the bound state Klein-Gordon equation for these exponential-type potentials. These methods include the supersymmetric (SUSY) and shape invariance method, the asymptotic iteration method (AIM), the Nikiforov-Uvarov (NU) Method (Louis *et al* 2015). The Klein-Gordon equation for the potential under studies is solved by using the parametric NU method to obtain the energy eigenvalues and eigen functions of the bound state.

The main aim of this paper is to use the proposed approximation and the Nikiforov-Uvarov method to obtain the bound state solutions of the Schrödinger equation with Woods-saxon (similar to the potential obtained by Louis *et al.,* 2017) plus attractive inversely quadratic (WSAIQ) potential defined as:

$$V(r)=\frac{-V_0}{1+e^{2\alpha r}}-\frac{V_0{}'}{r^2} \qquad (1)$$

where $V_0, V_0{}'$ are the strength of the potential for woods-saxon and inversely quadratic potential respectively. The rest of the paper is organised as follows. In section 2, the parametric Nikiforov-Uvarov method is presented. Factorization method is presented in section 3. In section 4, solutions of the radial part of Schrodinger equation with WSAIQ potential is presented. We discuss the results of our work in section 5. Finally, we present a brief conclusion in section 6.

2. REVIEW OF PARAMETRIC NIKIFAROV-UVAROV METHOD

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi_n'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \qquad (2)$$

where σ(s) and $\bar{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3]\Psi(s) = 0 \qquad (3)$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s),\ \sigma(s) = s(1-c_3 s),\ \bar{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3$$
(4)

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \qquad (5)$$

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \qquad (6)$$

while the wave function is given as:

$$\Psi_n(s) = N_{n,l} S^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1-2c_3 s) \qquad (7)$$

where

$$c_4 = \frac{1}{2}(1-c_1),\ c_5 = \frac{1}{2}(c_2 - 2c_3),\ c_6 = c_5^2 + \epsilon_1,\ c_7 = 2c_4 c_5 - \epsilon_2,\ c_8 = c_4^2 + \epsilon_3,$$
$$c_9 = c_3 c_7 + c_3^2 c_8 + c_6,\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c3\sqrt{c_8}\right)$$
$$c_{12} = c_4 + \sqrt{c_8},\ c_{13} = c_5 - \left(\sqrt{c_9} + c3\sqrt{c_8}\right) \qquad (8)$$

and P_n is the orthogonal polynomials.

Given that:

$$P_n^{(\alpha,\beta)} = \sum_{r=0}^{n} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\beta-r+1)(n-r)!r!}\left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \qquad (9)$$

This can also be expressed in terms of the Rodriguez's formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n\left((x-1)^{n+\alpha}(x+1)^{n+\beta}\right) \tag{10}$$

3. SOLUTION OF THE RADIAL KLEIN-GORDON EQUATION

The radial part of the Klein-Gordon Equation with vector V(r) potential = scalar S(r) potential in atomic units (ħ = c = 1) is given as:

$$\frac{d^2R(r)}{dr^2} + [(E^2 - M^2) - 2(E+M)V(r)]R(r) = 0 \tag{11}$$

Substituting potential of Eq. (1) into the Klein-Gordon equation of eq. (11), we obtain:

$$\frac{d^2R(r)}{dr^2} + \left[(E^2 - M^2) - 2(E+M)\left(\frac{V_0}{(1+e^{2\alpha r})} - \frac{V_0{}'}{r^2}\right)\right]R(r) = 0 \tag{12}$$

Since the Klein-Gordon equation with above combine potentials rarely has exact analytical solution, an approximation to the centrifugal term has to be made. The good approximation for $1/r^2$ in the centrifugal barrier is taken as:

$$\frac{1}{r^2} = \frac{4\alpha^2}{(1+e^{2\alpha r})^2}, \tag{13}$$

Similar to other related work (Bayrak and Sahin, 2015).

To solve Eq. (12) by the present method, we need to recast Eq. (13) and applying the transformation given as $s = -e^{2\alpha r}$.

$$R_{nl}''(s) + \frac{(1-s)}{(1-s)s}R_{nl}'(s) + \frac{1}{s^2(1-s)^2}\left[\left(\frac{E^2-M^2}{4\alpha^2}\right)s^2 + \left(-2\left(\frac{E^2-M^2}{4\alpha^2}\right) - 2\left(\frac{E+M}{4\alpha^2}\right)V_0\right)s + \frac{E^2-M^2}{4\alpha^2} + 2(E+M)V_0{}' + 2\left(\frac{E+M}{4\alpha^2}\right)V_0\right]R_{nl}(s) = 0 \tag{14}$$

$$let\ -\beta^2 = \frac{E^2-M^2}{4\alpha^2};\ B = 2\left(\frac{E+M}{4\alpha^2}\right)V_0;\ C = 2(E+M)V_0{}' \tag{15}$$

$$R_{nl}''(s) + \frac{(1-s)}{(1-s)s}R_{nl}'(s) + \frac{1}{s^2(1-s)^2}[-\beta^2 s^2 + (2\beta^2 - B)s + (-\beta^2 + B + C)]\,R_{nl}(s)=0 \tag{16}$$

$$R_{nl}''(s) + \frac{(1-s)}{(1-s)s}R_{nl}'(s) + \frac{1}{s^2(1-s)^2}[-\beta^2 s^2 + (2\beta^2 - B)s - (\beta^2 - B - C)]\,R_{nl}(s)=0 \tag{17}$$

where $\epsilon_1 = -\beta^2$; $\epsilon_2 = 2\beta^2 - B$; $\epsilon_3 = \beta^2 - B - C$

Comparing eqn. (4) with eqn. (17), we obtain the following parameters:

$$c_1 = c_2 = c_3 = 1,\ c_4 = 0,\ c_5 = -\frac{1}{2},\ c_6 = \frac{1}{4} + \beta^2,\ c_7 = (-2\beta^2 + B),\ c_8 = \beta^2 - B - C$$

$c_9 = \frac{1}{4} - C$, $c_{10} = 1 + 2\sqrt{\beta^2 - B - C}$, $c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} - C} + \sqrt{\beta^2 - B - C}\right)$,

$$c_{12} = \sqrt{\beta^2 - B - C},\ c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} - C} + \sqrt{\beta^2 - B - C}\right) \quad (18)$$

Substituting eqns. (18) into energy eigenvalues equation of eqn. (6) we obtain the energy equation for this system as

$$n^2 - (2n+1)\left(-\frac{1}{2}\right) + (2n+1)\left(\sqrt{\frac{1}{4} - C} + \sqrt{\beta^2 - B - C}\right) + B - 2\beta^2 + 2\beta^2 - 2B - 2C + 2\sqrt{(\beta^2 - B - C)(\tfrac{1}{4} - C)} = 0 \quad (19)$$

Solving eqn. (19) explicitly, we obtain the energy eigenvalues of the system as:

$$[E^2 - M^2] = -4\propto^2\left[\left(\frac{n^2 + n + \frac{1}{2} + \left(\frac{E+M}{2\alpha^2}\right)V_0 + 4(E+M)V_0' + (2n+1)\left(\sqrt{\frac{1}{4} - 2(E+M)V_0'}\right)}{(2n+1) + 2\sqrt{\frac{1}{4} - 2(E+M)V_0'}}\right)^2 + 2\left(\frac{E+M}{4\alpha^2}\right)V_0 + 2(E+M)V_0'\right] \quad (20)$$

Using eqns. (8) and (24), the wave function of this system is obtained as:

$$\Psi_{n,l}(s) = N_{n,l}\, S^{\epsilon}(1-S)^{\frac{1}{2}+\upsilon} P_n^{(2\epsilon, 2\upsilon)}(1-2s) \quad (21)$$

where $\epsilon = \sqrt{\beta^2 - B - C}$, and $\upsilon = \sqrt{\frac{1}{4} - C}$.

Furthermore, the relation between the hypergeometric function and the Jacobi polynomials are:

$$P_n^{(a,b)}(z) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)} \times {}_2F_1\left(-n, n+a+b+1; 1+a; \frac{1-z}{2}\right) \quad (22)$$

with $a = 2\epsilon > -1, b = 2\nu > -1$ under the transformation $z = (1-2s)$

The normalization constant $N_{n,l}$ can be found from normalization condition as:

$$\int_0^\infty |R(r)|^2\, dr = \alpha^{-1}\int_0^1 \frac{1}{s}\left|R_{n,l}(r)\right|^2 ds = 1 \quad (23)$$

By using the following integral formula:

$$\int_0^1 (1-z)^{2(\delta+1)} z^{2\lambda-1} \{F_1 - n, n+2(\delta+\lambda+1); 1+2\lambda; z\}^2 dz = \frac{(n+\delta+1)n!\Gamma(n+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\delta+\lambda+1)\Gamma(n+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n)} \quad (\text{if } z = 1-2s) \tag{24}$$

with the help of eqn. (24) and after some calculations the normalization constant $N_{n,l}$ is obtained as:

$$N_{n,l} = \sqrt{\frac{n!2\epsilon\left(n+\nu+\frac{1}{2}+\epsilon\right)\Gamma\left(2\left(\nu+\frac{1}{2}+\epsilon\right)+n\right)}{\alpha\left(n+\nu+\frac{1}{2}\right)\Gamma(n+2\epsilon+1)\Gamma(n+2\nu+1)}} \tag{25}$$

Finally, the total normalized wave function, $\psi(r,\theta,\phi)$ of the woods-saxon potential plus attractive inversely quadratic potential is obtained as:

$$\psi(r,\theta,\phi) = \frac{R(r)}{r} Y_{lm}(\theta,\phi)$$

$$\psi(r,\theta,\phi) = \sqrt{\frac{n!2\epsilon\left(n+\nu+\frac{1}{2}+\epsilon\right)\Gamma\left(2\left(\nu+\frac{1}{2}+\epsilon\right)+n\right)}{\alpha\left(n+\nu+\frac{1}{2}\right)\Gamma(n+2\epsilon+1)\Gamma(n+2\nu+1)}} \times \frac{1}{r}(-e^{2\alpha r})^{\epsilon}(1+e^{2\alpha r})^{\frac{1}{2}+\nu} \frac{\Gamma(n+2\epsilon+1)}{n!\Gamma(2\epsilon+1)} \times {}_2F_1\left(-n, n+2\epsilon+2\nu+1; 1+2\epsilon; \frac{1-z}{2}\right) Y(\theta,\phi) \tag{26}$$

4. DISCUSSION

Considering the proposed potential in equation (1), the radial Klein-gordon equation has been solved and the energy eigen values is obtained in equation (26) as:

$$[E^2 - M^2] = -4 \propto^2 \left[\left(\frac{n^2+n+\frac{1}{2}+\left(\frac{E+M}{2\alpha^2}\right)V_0 + 4(E+M){V_0}' + (2n+1)\left(\sqrt{\frac{1}{4}-2(E+M){V_0}'}\right)}{(2n+1)+2\sqrt{\frac{1}{4}-2(E+M){V_0}'}} \right)^2 + 2\left(\frac{E+M}{4\alpha^2}\right)V_0 + 2(E+M){V_0}' \right] \tag{20}$$

Case 1: If $V_0' = 0$ in equation (12), the potential turns back into the Woods-Saxon potential expressed as:

$$[E^2 - M^2] = -4 \propto^2 \left[\left(\frac{(n+1)^2 + \left(\frac{E+M}{2\alpha^2}\right)V_0}{2(n+1)} \right)^2 + \left(\frac{E+M}{2\alpha^2}\right)V_0 \right] \tag{27}$$

$$[E^2 - M^2] = -4 \propto^2 \left[\left(\frac{(E+M)V_0}{4\alpha(n+1)} \right)^2 + 3\left(\frac{E+M}{4\alpha^2}\right)V_0 + \left(\frac{n+1}{2}\right)^2 \right] \tag{28}$$

Eq. (28) is similar to the result obtained by Bayrak and Sahin, 2015.
Case 2: $if\ V_0 = 0$, the energy eigenvalues for attractive inversely quadratic potential becomes:

$$[E^2 - M^2] = -4 \propto^2 \left[\left(\frac{n^2+n+\frac{1}{2}+ 4(E+M){V_0}'+(2n+1)\left(\sqrt{\frac{1}{4}-2(E+M){V_0}'}\right)}{(2n+1)+2\sqrt{\frac{1}{4}-2(E+M){V_0}'}} \right)^2 + 2(E+M){V_0}' \right] \qquad (29)$$

The results obtained in eq. (29) is similar to the result obtained by Louis *et al.*, 2017

5. CONCLUSION

In this paper, we have obtained the bound state solution of the Klein-gordon equation with Woods-Saxon plus attractive inversely quadratic (WSAIQ) potential via parametric Nikiforov-Uvarov (NU) method. The energy eigenvalues and the corresponding total normalized wave functions expressed in terms of the hypergeometric functions for the system are also obtained.

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