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Perturbed moments of earnings per share in a random environment

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ABSTRACT

The perturbation ε of a random environment Ω is considered. There is proved that as $\varepsilon \to 0$ the perturbed third and the perturbed fourth moments differ from the third and the fourth moments respectively very little. The convergence rate of the perturbed third and the perturbed fourth moments to the unperturbed ones is investigated.

***Keywords*:** earnings per share, risk, third and fourth moments, perturbation of an environment, convergence rate

1. INTRODUCTION

A problem about finding of the expected profit and risk is sufficiently important. In particular, in (Sharpe 1964, p. 425-442) and (Sharpe 1970) are calculated the average expected returns and risks of individual securities and whole their portfolios. In articles (Yeleiko, Borotyuk 1999) the average expected earnings per share and risk of share are investigated. There is showed that the convergence rate of the perturbed profit and the perturbed risk to the unperturbed ones is linear. The necessary and sufficient conditions at which the convergence rate of the perturbed profit to unperturbed one has order $k \in N$ are established. The similar problem for the perturbed risk is considered.

For studying such characteristics of statistical distributions as asymmetric function and excess are used the third and the fourth moments respectively. It is vital to note that these characteristics are quite important in the formation of the portfolio. So investigation of the behavior of the third and the fourth moments in the perturbed environment is actual at this time.

In this paper we will consider the third and the fourth moments of earnings per share. We will clear up the matter about the change of these quantities at the perturbation of an environment. We will also investigate the convergence rate of the perturbed third and the perturbed fourth moments of earnings per share to the unperturbed ones.

2. BEHAVIOUR OF THE MOMENTS OF EARNINGS PER SHARE AT THE PERTURBATION OF AN ENVIRONMENT

Let us consider a share in a random environment Ω. Suppose that there are N methods of receiving earnings per share in the environment Ω. Denote by A_i an event which occurs if and only if earnings per share is received by i-th method, $A_i \bigcap A_j = \varnothing$, if $i \neq j$, and $\bigcap_{i=1}^{N} A_i = \Omega$. Let p_i be a probability of the event A_i, i.e. $p_i = P(A_i)$, and let r_i be earnings per share if the event A_i occured, $i = \overline{1, N}$. Define a random variable ξ which expresses earnings per share by $\xi(\omega) = r_i$, if $\omega \in A_i$. Then the average expected earnings per share, denoted by $\bar{r}$, is the mean of the random variable ξ. Clearly,

$$\bar{r} = \sum_{i=1}^{N} p_i r_i . \tag{1}$$

The risk of share, denoted by σ, is the deviation of the random variable ξ. Thus

$$\sigma = \sqrt{\sum_{i=1}^{N} p_i (r_i - \bar{r})^2} . \tag{2}$$

The third and the fourth moments of the random variable ξ are defined respectively by formulas

$$\overline{r^3} = \sum_{i=1}^{N} p_i r_i^3 ; \quad \overline{r^4} = \sum_{i=1}^{N} p_i r_i^4 . \tag{3}$$

Analyse as changing environment influences on change of earnings per share. Let us describe changing environment by some parameter ε. Such changing is called the perturbation of environment. The probability p_i and the earnings per share r_i, $i = \overline{1, N}$, will change in result such perturbation. Changed probability, denoted p_i^{ε}, is called the perturbed probability. Similarly changed profitability, denoted r_i^{ε}, is called the perturbed profitability. Assume that there exist limits

$$p_i = \lim_{\varepsilon \to 0} p_i^{\varepsilon} , \quad r_i = \lim_{\varepsilon \to 0} r_i^{\varepsilon} , \quad i = \overline{1, N}, \tag{4}$$

A sum of the perturbed probabilities must be equal to 1, i.e.

$$\sum_{i=1}^{N} p_i^{\varepsilon} = 1.$$

We suppose that at the perturbations of environment a number of events A_i remains unchanged.

In the article (Yeleiko, Borotyuk 1999, p. 95-98) is proved that at sufficiently small changes in the environment the average expected earnings per share differs from the average expected earnings per share in unchanged environment very little. Thus there exists limit

$$\bar{r} = \lim_{\varepsilon \to 0} \overline{r^{\varepsilon}},$$

where $\overline{r^{\varepsilon}}$ – the perturbed average expected earnings per share is defined as

$$\overline{r^{\varepsilon}} = \sum_{i=1}^{N} p_i^{\varepsilon} r_i^{\varepsilon}. \qquad (5)$$

Similarly, there is showed that the perturbed risk of share, denoted by σ^{ε}, tends to σ as $\varepsilon \to 0$. The perturbed risk of share is defined as

$$\sigma^{\varepsilon} = \sqrt{\sum_{i=1}^{N} p_i^{\varepsilon} (r_i^{\varepsilon} - \overline{r^{\varepsilon}})^2}. \qquad (6)$$

Consequently, at sufficiently small changes in the environment the risk of the security has rather small deviation from the risk of the security in unchanged environment.

Let us elucidate as the third and the fourth moments will change at the perturbations of environment. By $\overline{r_{\varepsilon}^3}$ and $\overline{r_{\varepsilon}^4}$ denote the perturbed third and the perturbed fourth moments respectively. These moments are defined as

$$\overline{r_{\varepsilon}^3} = \sum_{i=1}^{N} p_i^{\varepsilon} (r_i^{\varepsilon})^3; \quad \overline{r_{\varepsilon}^4} = \sum_{i=1}^{N} p_i^{\varepsilon} (r_i^{\varepsilon})^4. \qquad (7)$$

The answer to this question is given by the following theorem.

Theorem 1.

Let conditions (4) hold. Then the perturbed third and the perturbed fourth moments, defined by (7), tend to the third and the fourth moments, defined by (3), respectively as $\varepsilon \to 0$.

P r o o f. From (4) it follows that for all $i = \overline{1, N}$

$$\lim_{\varepsilon \to 0} p_i^{\varepsilon} (r_i^{\varepsilon})^3 = p_i r_i^3.$$

Since the sum in the equalities (7) is finite, we obtain

$$\lim_{\varepsilon\to 0}\overline{r_\varepsilon^3}=\sum_{i=1}^{N}\lim_{\varepsilon\to 0}p_i^\varepsilon(r_i^\varepsilon)^3=\sum_{i=1}^{N}p_i r_i^3=\overline{r^3}.$$

Likewise we can prove that

$$\lim_{\varepsilon\to 0}\overline{r_\varepsilon^4}=\overline{r^4}.$$

The theorem is proved. □

3. THE CONVERGENCE RATE OF THE PERTURBED THIRD AND THE PERTURBED FOURTH MOMENTS TO THE UNPERTURBED ONES

Let p_i^ε and r_i^ε be expressed as

$$p_i^\varepsilon = p_i + \lambda_{1i}\varepsilon + \ldots + \lambda_{li}\varepsilon^l + o(\varepsilon^l), \tag{8}$$

$$r_i^\varepsilon = r_i + \mu_{1i}\varepsilon + \ldots + \mu_{li}\varepsilon^l + o(\varepsilon^l), \tag{9}$$

where $l \in N$, $\lambda_{1i},\ldots,\lambda_{li},\mu_{1i},\ldots,\mu_{li}$, $i=\overline{1,N}$, – some constants, $\varepsilon > 0$.

Now we will give the following definition.

Definition 1. *The rate of convergence* $f(\varepsilon)\underset{\varepsilon\to 0}{\to} a$ *has order* $k \in N$ *if*

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)-a}{\varepsilon^k}=C,$$

where C – some constant. In case $k=1$ the convergence rate is called linear.

In the article (Yeleiko, Borotyuk 1999, p. 95-98) is showed that the convergence rate of the perturbed profit (5) and the perturbed risk (6) to the unperturbed ones (1), (2) is linear. There are established the necessary and sufficient conditions at which the convergence rate of the perturbed profit (5) to unperturbed one (1) has order $k+1(k=1,\ldots,l-1)$, where l is taken from (8), (9). There are also established the necessary and sufficient conditions at which the convergence rate of the perturbed risk (6) to unperturbed one (2) has order $s(s\le l)$, where l is taken from (8), (9). Find out the convergence rate of the perturbed third and the perturbed fourth moments to the unperturbed ones.

Theorem 2.

1. *The convergence rate of the perturbed third moment to the unperturbed one is linear.*
2. *The convergence rate of the perturbed third moment to unperturbed one has order* $t+1(t=1,\ldots,l-1)$ *if and only if the following conditions hold:*

$$\begin{cases} \sum_{i=1}^{N}(r_i^3\lambda_{1i}+3r_i^2 p_i\mu_{1i})=0, \\ \sum_{i=1}^{N}(r_i^3\lambda_{2i}+3r_i^2 p_i\mu_{2i}+3r_i^2\mu_{1i}\lambda_{1i}+3r_i p_i\mu_{1i}^2)=0, \\ \sum_{i=1}^{N}(r_i^3\lambda_{3i}+3r_i^2 p_i\mu_{3i}+3r_i^2\mu_{2i}\lambda_{1i}+3r_i^2\mu_{1i}\lambda_{2i}+3r_i\lambda_{1i}\mu_{1i}^2+6r_i p_i\mu_{1i}\mu_{2i}+p_i\mu_{1i}^3)=0, \\ \dots \\ \sum_{i=1}^{N}(r_i^3\lambda_{ti}+3r_i^2 p_i\mu_{ti}+\sum_{\substack{m,n,k,q=0 \\ m+n+k+q=t}}^{t-1}\mu_{mi}\mu_{ni}\mu_{ki}\lambda_{qi})=0. \end{cases} \tag{10}$$

P r o o f.

Consider the perturbed third moment $\overline{r_\varepsilon^3}=\sum_{i=1}^{N}p_i^\varepsilon(r_i^\varepsilon)^3$. From (8), (9) it follows that

$$\overline{r_\varepsilon^3}=\sum_{i=1}^{N}(p_i+\lambda_{1i}\varepsilon+\ldots+\lambda_{li}\varepsilon^l+o(\varepsilon^l))(r_i+\mu_{1i}\varepsilon+\ldots+\mu_{li}\varepsilon^l+o(\varepsilon^l))^3.$$

First calculate the value $(r_i^\varepsilon)^2$:

$$(r_i+\mu_{1i}\varepsilon+\mu_{2i}\varepsilon^2+\ldots+\mu_{li}\varepsilon^l+o(\varepsilon^l))^2=r_i^2+\mu_{1i}^2\varepsilon^2+\mu_{2i}^2\varepsilon^4\ldots+\mu_{li}^2\varepsilon^{2l}+$$

$$+2r_i\mu_{1i}\varepsilon+2r_i\mu_{2i}\varepsilon^2+\ldots+2r_i\mu_{li}\varepsilon^l+2\mu_{1i}\mu_{2i}\varepsilon^3+2\mu_{1i}\mu_{3i}\varepsilon^4+\ldots+2\mu_{1i}\mu_{li}\varepsilon^{l+1}+$$

$$+2\mu_{2i}\mu_{3i}\varepsilon^5+\ldots+2\mu_{2i}\mu_{li}\varepsilon^{l+2}+\ldots+2\mu_{l-1,i}\mu_{li}\varepsilon^{2l-1}+o(\varepsilon^l).$$

By μ_{0i} denote r_i. Then

$$(r_i^\varepsilon)^2=\sum_{\substack{m,n=0 \\ m+n\le l}}^{l}\mu_{mi}\mu_{ni}e^{m+n}+o(\varepsilon^l).$$

Hence, we obtain

$$(r_i^{\varepsilon})^3 = \left(\sum_{\substack{m,n=0 \\ m+n\le l}}^{l} \mu_{mi}\mu_{ni}e^{m+n} + o(\varepsilon^l) \right)\left(\sum_{k=0}^{l} \mu_{ki}e^{k} + o(\varepsilon^l) \right) = \sum_{\substack{m,n,k=0 \\ m+n+k\le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}e^{m+n+k} + o(\varepsilon^l). \quad (11)$$

Therefore, denoting by λ_{0i} the probability p_i, we receive

$$\overline{r_{\varepsilon}^3} = \sum_{i=1}^{N} \left(\sum_{\substack{m,n,k=0 \\ m+n+k\le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}e^{m+n+k} + o(\varepsilon^l) \right)\left(\sum_{q=0}^{l} \lambda_{qi}\varepsilon^{q} + o(\varepsilon^l) \right) =$$

$$= \sum_{i=1}^{N} \sum_{\substack{m,n,k,q=0 \\ m+n+k+q\le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\lambda_{qi}e^{m+n+k+q} + o(\varepsilon^l) = \sum_{i=1}^{N} \mu_{0i}^3 \lambda_{0i} +$$

$$+ \sum_{i=1}^{N} \sum_{\substack{m,n,k,q=0 \\ 1\le m+n+k+q\le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\lambda_{qi}e^{m+n+k+q} + o(\varepsilon^l). \quad (12)$$

Note that $\mu_{0i}^3 \lambda_{0i} = r_i^3 p_i$. Thus,

$$\overline{r_{\varepsilon}^3} = \overline{r^3} + \sum_{i=1}^{N} \sum_{\substack{m,n,k,q=0 \\ 1\le m+n+k+q\le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\lambda_{qi}e^{m+n+k+q} + o(\varepsilon^l).$$

Consequently,

$$\overline{r_{\varepsilon}^3} - \overline{r^3} = \sum_{i=1}^{N} (r_i^3 \lambda_{1i} + 3r_i^2 p_i \mu_{1i})\varepsilon + \sum_{i=1}^{N} (r_i^3 \lambda_{2i} + 3r_i^2 p_i \mu_{2i} + 3r_i^2 \mu_{1i}\lambda_{1i} + 3r_i p_i \mu_{1i}^2)\varepsilon^2 +$$

$$+ \sum_{i=1}^{N} (r_i^3 \lambda_{3i} + 3r_i^2 p_i \mu_{3i} + 3r_i^2 \mu_{2i}\lambda_{1i} + 3r_i^2 \mu_{1i}\lambda_{2i} + 3r_i \lambda_{1i}\mu_{1i}^2 + 6r_i p_i \mu_{1i}\mu_{2i} + p_i \mu_{1i}^3)\varepsilon^3 + \ldots +$$

$$+\sum_{i=1}^{N}\left(r_i^3\lambda_{li} + 3r_i^2 p_i\mu_{li} + \sum_{\substack{m,n,k,q=0 \\ m+n+k+q=l}}^{l-1} \mu_{mi}\mu_{ni}\mu_{ki}\lambda_{qi} \right)\varepsilon^l + o(\varepsilon^l). \tag{13}$$

Whence as $\varepsilon \to 0$ we have

$$\overline{r_\varepsilon^3} - \overline{r^3} = \sum_{i=1}^{N}(r_i^3\lambda_{1i} + 3r_i^2 p_i\mu_{1i})\varepsilon + o(\varepsilon),$$

that is, by the definition 1 the rate of convergence $\overline{r_\varepsilon^3} \underset{\varepsilon\to 0}{\to} \overline{r^3}$ is linear. From (13) it follows that the rate of convergence $\overline{r_\varepsilon^3} \underset{\varepsilon\to 0}{\to} \overline{r^3}$ will have order 2, if the following condition holds:

$$\sum_{i=1}^{N}(r_i^3\lambda_{1i} + 3r_i^2 p_i\mu_{1i}) = 0.$$

Similarly, the rate of convergence $\overline{r_\varepsilon^3} \underset{\varepsilon\to 0}{\to} \overline{r^3}$ will have order $t+1(t=1,\ldots,l-1)$, if the conditions (10) hold.

Obviously, if the conditions (10) hold, the rate of convergence $\overline{r_\varepsilon^3} \underset{\varepsilon\to 0}{\to} \overline{r^3}$ will have order $t+1(t=1,\ldots,l-1)$.

The theorem is proved. □

Theorem 3.

1. *The convergence rate of the perturbed fourth moment to the unperturbed one is linear.*
2. *The convergence rate of the perturbed fourth moment to unperturbed one has order* $t+1(t=1,\ldots,l-1)$ *if and only if the following conditions hold:*

$$\begin{cases} \sum_{i=1}^{N} (r_i^4 \lambda_{1i} + 4r_i^3 p_i \mu_{1i}) = 0, \\ \sum_{i=1}^{N} (r_i^4 \lambda_{2i} + 4r_i^3 p_i \mu_{2i} + 4r_i^3 \mu_{1i} \lambda_{1i} + 6r_i^2 p_i \mu_{1i}^2) = 0, \\ \sum_{i=1}^{N} (r_i^4 \lambda_{3i} + 4r_i^3 p_i \mu_{3i} + 4r_i^3 \mu_{2i} \lambda_{1i} + 4r_i^3 \mu_{1i} \lambda_{2i} + 5r_i^2 \lambda_{1i} \mu_{1i}^2 + 12r_i^2 p_i \mu_{1i} \mu_{2i} + 4r_i p_i \mu_{1i}^3) = 0, e(e) \\ \dots \\ \sum_{i=1}^{N} (r_i^4 \lambda_{li} + 4r_i^3 p_i \mu_{li} + \sum_{\substack{m,n,k,q,f=0 \\ m+n+k+q+f=t}}^{t-1} \mu_{mi} \mu_{ni} \mu_{ki} \mu_{fi} \lambda_{qi}) = 0. \end{cases} \tag{14}$$

P r o o f.

Consider the perturbed fourth moment $\overline{r_\varepsilon^4} = \sum_{i=1}^{N} p_i^\varepsilon (r_i^\varepsilon)^4$. Since $p_i^\varepsilon (r_i^\varepsilon)^4 = [p_i^\varepsilon (r_i^\varepsilon)^3] r_i^\varepsilon$, from (8), (9), (11) it follows that

$$p_i^\varepsilon (r_i^\varepsilon)^4 = \left[\left(\sum_{q=0}^{l} \lambda_{qi} \varepsilon^q + o(\varepsilon^l) \right) \left(\sum_{\substack{m,n,k=0 \\ m+n+k \le l}}^{l} \mu_{mi} \mu_{ni} \mu_{ki} e^{m+n+k} + o(\varepsilon^l) \right) \right] \left(\sum_{f=0}^{l} \mu_{fi} \varepsilon^f + o(\varepsilon^l) \right).$$

Likewise to (12)

$$p_i^\varepsilon (r_i^\varepsilon)^4 = \left(\sum_{\substack{m,n,k,q=0 \\ m+n+k+q \le l}}^{l} \mu_{mi} \mu_{ni} \mu_{ki} \lambda_{qi} e^{m+n+k+q} + o(\varepsilon^l) \right) \left(\sum_{f=0}^{l} \mu_{fi} \varepsilon^f + o(\varepsilon^l) \right) =$$

$$= \sum_{\substack{m,n,k,q,f=0 \\ m+n+k+q+f \le l}}^{l} \mu_{mi} \mu_{ni} \mu_{ki} \mu_{fi} \lambda_{qi} e^{m+n+k+q+f} + o(\varepsilon^l).$$

So we obtain

$$\overline{r_\varepsilon^4} = \sum_{i=1}^{N} \left(\sum_{\substack{m,n,k,q,f=0 \\ m+n+k+q+f \le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\mu_{fi}\lambda_{qi} e^{m+n+k+q+f} + o(\varepsilon^l) \right) =$$

$$= \sum_{i=1}^{N} p_i r_i^4 + \sum_{i=1}^{N} \sum_{\substack{m,n,k,q,f=0 \\ 1 \le m+n+k+q+f \le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\mu_{fi}\lambda_{qi} e^{m+n+k+q+f} + o(\varepsilon^l).$$

Hence

$$\overline{r_\varepsilon^4} - \overline{r^4} = \sum_{i=1}^{N} \sum_{\substack{m,n,k,q,f=0 \\ 1 \le m+n+k+q+f \le l}}^{l} \mu_{mi}\mu_{ni}\mu_{ki}\mu_{fi}\lambda_{qi} e^{m+n+k+q+f} + o(\varepsilon^l) =$$

$$= \sum_{i=1}^{N} (r_i^4 \lambda_{1i} + 4r_i^3 p_i \mu_{1i})\varepsilon + \sum_{i=1}^{N} (r_i^4 \lambda_{2i} + 4r_i^3 p_i \mu_{2i} + 4r_i^3 \mu_{1i}\lambda_{1i} + 6r_i^2 p_i \mu_{1i}^2)\varepsilon^2 +$$

$$+ \sum_{i=1}^{N} (r_i^4 \lambda_{3i} + 4r_i^3 p_i \mu_{3i} + 4r_i^3 \mu_{2i}\lambda_{1i} + 4r_i^3 \mu_{1i}\lambda_{2i} + 5r_i^2 \lambda_{1i}\mu_{1i}^2 + 12r_i^2 p_i \mu_{1i}\mu_{2i} + 4r_i p_i \mu_{1i}^3)\varepsilon^3 + \ldots +$$

$$+ \sum_{i=1}^{N} \left(r_i^4 \lambda_{li} + 4r_i^3 p_i \mu_{li} + \sum_{\substack{m,n,k,q,f=0 \\ m+n+k+q+f = l}}^{l-1} \mu_{mi}\mu_{ni}\mu_{ki}\mu_{fi}\lambda_{qi} \right) \varepsilon^l + o(\varepsilon^l). \tag{15}$$

Thus, as $\varepsilon \to 0$ we receive

$$\overline{r_\varepsilon^4} - \overline{r^4} = \sum_{i=1}^{N} (r_i^4 \lambda_{1i} + 4r_i^3 p_i \mu_{1i})\varepsilon + o(\varepsilon),$$

that is, by the definition 1 the rate of convergence $\overline{r_\varepsilon^4} \underset{\varepsilon \to 0}{\to} \overline{r^4}$ is linear. From (15) it follows that the rate of convergence $\overline{r_\varepsilon^4} \underset{\varepsilon \to 0}{\to} \overline{r^4}$ will have order 2, if the following condition holds:

$$\sum_{i=1}^{N}(r_i^4 \lambda_{1i} + 4r_i^3 p_i \mu_{1i}) = 0.$$

Similarly, the rate of convergence $\overline{r_\varepsilon^4} \underset{\varepsilon \to 0}{\to} \overline{r^4}$ will have order $t+1 (t=1,\ldots,l-1)$, if the conditions (14) hold.

Obviously, if the conditions (14) hold, the rate of convergence $\overline{r_\varepsilon^4} \underset{\varepsilon \to 0}{\to} \overline{r^4}$ will have order $t+1 (t=1,\ldots,l-1)$.

The theorem is proved. □

4. CONCLUSIONS

In this paper the third and the fourth moments of earnings per share are considered. Here is proved that these quantities are hardly changing at the perturbation of an environment. The convergence rate of the perturbed third and the perturbed fourth moments to the unperturbed ones is investigated.

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