Hamiltonian cycle containing selected sets of edges of a graph

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ABSTRACT

The aim of this paper is to characterize for every \( k \geq 1 \) all \( (l+3) \)-connected graphs \( G \) on \( n \geq 3 \) vertices satisfying \( P(n+k) \):

\[
d_{G}(x,y) = 2 \Rightarrow \max\{d(x,G), d(y,G)\} \geq \frac{n+k}{2}
\]

for each pair of vertices \( x \) and \( y \) in \( G \), such that there is a path system \( S \) of length \( k \) with \( l \) internal vertices which components are paths of length at most 2 satisfying:

\[
P: \ u_1u_2u_3 \subset S \text{ and } d(u_1,G), d(u_2,G) \geq \frac{n+k}{2} \Rightarrow d(u_3,G) \geq \frac{n+k}{2}, \text{ such that } S \text{ is not contained in any hamiltonian cycle of } G.
\]

Keywords: Cycle; hamiltonian cycle; matching; path

1. INTRODUCTION

We consider only finite graphs without loops and multiple edges. By \( V \) or \( V(G) \) we denote the vertex set of graph \( G \) and respectively by \( E \) or \( E(G) \) the edge set of \( G \). By \( d(x,G) \) or \( d(x) \) we denote the degree of a vertex \( x \) in the graph \( G \) and by \( d(x,y) \) or \( dG(x,y) \) the distance between \( x \) and \( y \) in \( G \).
**Definition 1.1.** (cf [7]) Let \( k, s_1, \ldots, s_l \) be positive integers. We call \( S \) a path system of length \( k \) if the connected components of \( S \) are paths:

\[
P^i : \quad x_0^i x_1^i \ldots x_{s_i}^i,
\]

And \( \sum_{i=1}^l s_i = k \).

Let \( S \) be a path system of length \( k \) and let \( x \in V(S) \). We shall call \( x \) an internal vertex if \( x \) is an internal vertex (cf [2]) in one of the paths \( P^1, \ldots, P^l \).

If \( q \) denotes the number of internal vertices in a path system \( S \) of length \( k \) then \( 0 \leq q \leq k - 1 \). If \( q = 0 \) then \( S \) is a \( k \)-matching (i.e. a set of \( k \) independent edges).

Let \( G \) be a graph and let \( S \) be a path system of length \( k \) in \( G \). Let paths \( P^1 : x_0^1 x_1^1 \ldots x_{s_1}^1, \ldots, P^l : x_0^l x_1^l \ldots x_{s_l}^l \) be components of \( S \). We can define a new graph \( \tilde{G} \) and a matching \( M_S \) in

\[
M_S = \{xy : x = x_0^i, y = x_{s_i}^i, i = 1, \ldots, l\}
\]

\[
V(\tilde{G}) = V(G) \setminus \bigcup_{i=1}^l \{x_1^i, \ldots, x_{s_i-1}^i\}
\]

\[
E(\tilde{G}) = M_S \cup \{xy : x, y \in V(\tilde{G}) \text{ and } xy \in E(G)\}
\]

Let \( H \) be a subgraph or a matching of \( G \). By \( G \setminus H \) we denote the graph obtained from \( G \) by the deletion of the edges of \( H \).

**Definition 1.2.** \( F \) is an \( H \)-edge cut-set of \( G \) if and only if \( F \subset E(H) \) and \( G \setminus F \) is not connected.

**Definition 1.3.** \( F \) is said to be a minimal \( H \)-edge cut-set of \( G \) if and only if \( F \) is an \( H \)-edge cut-set of \( G \) which has no proper subset being an edge cut-set of \( G \).

**Definition 1.4.** (cf [5]) Let \( G \) be a graph on \( n \geq 3 \) vertices and \( k \geq 0 \). \( G \) is \( k \)-edge-hamiltonian if for every path system \( P \) of length at most \( k \) there exists a hamiltonian cycle of \( G \) containing \( P \).

Let \( G \) be a graph and \( H \subset G \) a subgraph of \( G \). For a vertex \( x \in V(G) \) we define the set \( N_H(x) = \{y \in V(H) : xy \in E(G)\} \). Let \( H \) and \( D \) be two subgraphs of \( G \). \( E(D,H) = \{xy \in E(G) : x \in V(D) \text{ and } y \in V(H)\} \). For a set of vertices \( A \) of a graph \( G \) we define the graph \( G(A) \) as the subgraph induced in \( G \) by \( A \).

In the proof we will only use oriented cycles and paths. Let \( C \) be a cycle and \( x \in V(C) \), then \( \bar{x} \) is the predecessor of \( x \) and \( x^+ \) is its successor. We denote the number of components of a graph \( G \) by \( \omega(G) \).
Definition 1.5. (cf [1]) Let $W$ be a property defined for all graphs of order $n$ and let $k$ be a non-negative integer. The property $W$ is said to be $k$-stable if whenever $G + xy$ has property $W$ and $d(x, G) + d(y, G) \geq k$ then $G$ itself has property $W$.

J.A. Bondy and V. Chvátal [1] proved the following theorem, which we shall need in the proof of our main result:

Theorem 1.1. Let $n$ and $k$ be positive integers with $k \leq n - 3$. Then the property of being $k$-edge-hamiltonian is $(n + k)$-stable.

In 1960 O. Ore [6] proved the following:

Theorem 1.2. Let $G$ be a graph on $n \geq 3$ vertices. If for all nonadjacent vertices $x, y \in V(G)$ we have
\[
d(x, G) + d(y, G) \geq n
\]
then $G$ is hamiltonian.

Geng-Hua Fan [3] has shown:

Theorem 1.3. Let $G$ be a 2-connected graph on $n \geq 3$ vertices. If $G$ satisfies
\[
P(n) : \quad d_G(x, y) = 2 \Rightarrow \max\{d(x, G), d(y, G)\} \geq \frac{n}{2}
\]
for each pair of vertices $x$ and $y$ in $G$, then $G$ is hamiltonian.

The condition for degree sum in Theorem 1.2 is called an Ore condition or an Ore type condition for graph $G$ and the condition $P(k)$ is called a Fan condition or a Fan type condition for graph $G$.

Later many Fan type theorems and Ore type theorems has been shown.

Now we shall present Las Vargnas [8] condition $L_{n,s}$.

Definition 1.6. Let $G$ be graph on $n \geq 2$ vertices and let $s$ be an integer such that $0 \leq s \leq n$. $G$ satisfies Las Vargnas condition $L_{n,s}$ if there is an arrangement $x_1, \ldots, x_n$ of vertices of $G$ such that for all $i, j$ if
\[
1 \leq i < j \leq n, \quad i + j \geq n - s, \quad x_ix_j \notin E(G),
\]
then $d(x_i, G) \leq i + s$ and $d(x_j, G) \leq j + s - 1$.

Las Vargnas [8] proved the following theorem:

Theorem 1.4. Let $G$ be a graph on $n \geq 3$ vertices and let $0 \leq s \leq n - 1$. If $G$ satisfies $L_{n,s}$ then $G$ is $s$-edge hamiltonian.
Note that condition $L_{n,s}$ is weaker than Ore condition.
Later Skupień and Wojda proved that the condition $L_{n,s}$ is sufficient for a graph to have a stronger property (for details see [7]). Wojda [9] proved the following Ore type theorem:

**Theorem 1.5** Let $G$ be a graph on $n \geq 3$ vertices, such that for every pair of nonadjacent vertices $x$ and $y$
\[
d(x, G) + d(y, G) > \frac{4n - 4}{3}.
\]
Then every matching of $G$ lies in a hamiltonian cycle.

In 1996 G. Gancarzewicz and A. P. Wojda [4] proved the following Fan type theorem:

**Theorem 1.6.** Let $G$ be a 3-connected graph of order $n \geq 3$ and let $M$ be a $k$-matching in $G$. If $G$ satisfies $P(n + k)$:
\[
d(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n + k}{2}
\]
for each pair of vertices $x$ and $y$ in $G$, then $M$ lies in a hamiltonian cycle of $G$ or $G$ has a minimal odd $M$-edge cut-set.

In this paper we shall find a Fan type condition under which every path system of length $k$ in a graph $G$ lies in a hamiltonian cycle.

For notation and terminology not defined above a good reference should be [2].

2. RESULT

**Theorem 2.1.** Let $G$ be a graph on $n \geq 3$ vertices and let $S$ be a path system of length $k$ with $l$ internal vertices which components are paths of length at most 2 such that if $P_1u_2u_3 \subset S$ and $d(u_1, G) \geq \frac{n + k}{2}$ then $d(u_3, G) \geq \frac{n + k}{2}$. If $G$ is $(l + 3)$-connected and $G$ satisfies $P(n + k)$:
\[
d_G(x, y) = 2 \Rightarrow \max\{d(x, G), d(y, G)\} \geq \frac{n + k}{2}
\]
for each pair of vertices $x$ and $y$ in $G$, then $S$ lies in a hamiltonian cycle of $G$ or the graph $\tilde{G}$ has a minimal odd $M_S$-edge cut-set.

Note that under assumptions of Theorem 2.1 we have $0 \leq l \leq \lfloor \frac{n}{2} \rfloor$.

It is clear that Theorem 1.6 is a simple consequence of Theorem 2.1.

3. PROOF

**Proof of Theorem 2.1.**
Consider $G$ and $S$ as in the assumptions of Theorem 2.1.

We can now define the set $A$:
\[ A = \{ x \in V(G) : d(x, G) \geq \frac{n+k}{2} \}. \]

Note that if \( x \) and \( y \) are nonadjacent vertices of \( A \) then the graph obtained from \( G \) by the addition of the edge \( xy \) also satisfies the assumptions of the theorem. Therefore and by Theorem 1.1 we may assume that:

\[ xy \in E(G) \quad \text{for any} \quad x, y \in A \quad \text{and} \quad x \neq y. \]  \hspace{1cm} (3.1)

By (3.1) \( A \) induces a complete subgraph \( G(A) \) of the graph \( G \). Let \( GV \setminus A \) be a graph obtained from \( G \) by deletion of vertices of the graph \( G(A) \) (i.e. vertices from the set \( A \)).

Now consider a component \( D \) of the graph \( GV \setminus A \).

Suppose that there exist two nonadjacent vertices in \( D \). Since \( D \) is connected we have two vertices \( x \) and \( y \) in \( D \) such that \( dG(x,y) = 2 \) and by the assumption that \( G \) satisfies \( P(n+k) \) we have \( x \in A \) or \( y \in A \), a contradiction.

So we can assume that every component of \( GV \setminus A \) is a complete graph \( K_\iota \), \( \iota \in I \), joined with \( G(A) \) by at least \( l + 3 \) edges.

If \( K_{i_0}, K_{i_1} \in \{ K_\iota \}_{\iota \in I} \) are such that \( i_0 \neq i_1 \) then:

\[ N(K_{i_0}) \cap N(K_{i_1}) = \emptyset. \]  \hspace{1cm} (3.2)

In fact, suppose that \( N(K_{i_0}) \cap N(K_{i_1}) \neq \emptyset \). Then we have a vertex \( y \in K_{i_0} \) and a vertex \( y' \in K_{i_1} \) such that \( dG(y,y') = 2 \) and by \( P(n+k) \) either \( y \in A \) or \( y' \in A \). This contradicts the fact that \( K_{i_0} \) and \( K_{i_1} \) are two connected components of \( GV \setminus A \).

If \( C \subseteq G \) is a cycle in \( G \) then be \( GV \setminus C \) we denote a graph obtained from \( G \) by deletion of vertices of the cycle \( C \).

The graph \( G \) consists of a complete graph \( GV \setminus A \) and of a family of complete components \( \{ K_\iota \}_{\iota \in I} \).

Let \( K \in \{ K_\iota \}_{\iota \in I} \).

Let \( P : u_1u_2u_3 \) be a path of length 2 from \( S \). \( P \) is called a \( A \)-ear if \( u_1, u_3 \in A \) and \( u_2 \in V(K) \), and respectively a \( K \)-ear if \( u_1, u_3 \in V(K) \) and \( u_2 \notin V(K) \) (in this case \( u_2 \in A \)).

If \( E(K,A) \cap E(S) \neq \emptyset \), then in \( E(K,A) \cap E(S) \) we can have a family of ears and a number of edges from \( E(S) \) which does not form any ear.

Now we shall define a cycle \( C \). First consider a path containing only all \( A \)-ears. Next we add to this path all remaining vertices from \( A \) and all edges from the set \( E(S) \cap E(G(A)) \). All those edges and vertices form the cycle \( C \).

Note that the cycle \( C \) performs the following conditions:

\[ \begin{align*}
   & \cdot C \text{ contains all edges of } E(S) \cap E(GV \setminus A) \text{ and all vertices of } A. \hspace{1cm} (3.3) \\
   & \cdot \text{If } K_{i_0} \text{ and } K_{i_1} \text{ are two different components of } GV \setminus C \text{ then} \\
   & \quad N(K_{i_0}) \cap N(K_{i_1}) = \emptyset. \hspace{1cm} (3.4) \\
   & \cdot \text{Let } x \notin V(C), y \in V(C) \text{ and } xy \in E(G) \text{ then:} \\
   & \quad \text{if } y \text{ is not an internal vertex of } S, \text{ then } y \in A, \\
   & \quad \text{if } y^- \text{ is not an internal vertex of } S, \text{ then } y \in A, \\
\end{align*} \]
if \( y^+ \) is not an internal vertex of \( S \), then \( y^+ \in A \).

Such cycle exists since \( GV \setminus A \) is a complete graph and \( G \) satisfies (3.2).

Hence \( G \) is \((l + 3)\)-connected, every component of \( GV \setminus C \) is a complete graph joined with \( C \) by at least 3 edges which ends are not internal vertices of \( S \).

Let \( K \) be a connected component of \( GV \setminus C \).

We shall show that we can extend the cycle \( C \) over all vertices of \( K \), over all edges of \( S \) in \( K \) and over all edges of \( S \) joining \( K \) with \( C \) preserving the properties (3.3) — (3.5) or that the graph \( \tilde{G} \) has a minimal odd \( M_S \)-edge cut-set.

**Case 1**

Among the edges joining \( K \) with \( C \) there are no edges from path system \( S \). Since \( G \) is \((l + 3)\)-connected \( K \) is joined with \( C \) by at least 3 edges which ends are not internal vertices of \( S \).

We have \( x_i y_i \) such that \( x_i \in K, y_i \in C \) and \( y_i^-, y_i^+ \in A \), for \( i = 1, 2, 3 \). We can assume that vertices \( x_1, x_2 \), are joined by one path \( P \) from \( S \). Here \( P \) is directed from \( x_2 \) to \( x_1 \).

![Figure (3.1).](image)

Suppose that also \( y_1 y_1^+, y_2 y_2^+ \in E(S) \). If \( y_3 y_3^+ \in E(S) \) (then \( y_3 \) is a start vertex of one path from \( S \) directed towards \( x_1 \)) we can consider the cycle (see Figure (3.1)):

\[
C' : x_3 v_1 \ldots v_k P y_1 y_1^+ \ldots y_2^- y_2^+ \ldots y_3^- y_1^- \ldots y_3^+ ,
\]

(3.6)

where \( v_1 \ldots v_k \) is a path containing all remaining vertices from the set

\[
V(K) \setminus (P \cup \{x_3\})
\]

and edges from the set \((E(S) \cap E(K)) \setminus E(P)\).
when \( y_3 \) \( y_3 \in E(S) \) we can carry out similar construction of cycle \( C' \).

\[
C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^+ \ldots y_2^- y_2 y_1^- \ldots y_3^- y_2 \ldots y_3^- .
\]

Note that we can do the same if \( y_1 \) and \( y_2 \) joined by one path from \( S \).
If \( y_2 \) and \( y_3 \) are end vertices of the same path from \( S \) or \( y_2 y_2^+ , y_3 y_3^+ \in E(S) \) we can carry out a similar construction.

Suppose that \( y_1 y_1^+ \in E(S) \). We can consider the cycle:

\[
C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^+ \ldots y_2^- y_2 y_1^- \ldots y_3^- y_2 \ldots y_3^- .
\]

Supposing that \( y_1 y_1, y_2 y_3 \in E(S) \) a good extension of \( C \) should be the cycle:

\[
C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^- \ldots y_3^- y_1^- \ldots y_2^- y_2 \ldots y_3^- .
\]

The last two cycles are good also if \( y_2 \) and \( y_3 \) are end vertices of the same path from \( S \).
when \( y_1 y_1^+ \in E(S) \) for \( i = 1, \ldots, 3 \) we can define \( C' \) as in (3.6).

It is clear that the new cycle \( C' \) fulfills (3.3) — (3.5) and is an extension of \( C \) such that

\[
V(C) \subset V(C') \quad \text{and} \quad ((E(C) \cup E(K)) \cap S), \quad (E(C, K) \cap S) \subset E(C'). \quad (3.7)
\]

If among the edges joining \( K \) with \( C \) there are no edges from path system \( S \) then all other situations can be reduced to those presented above.

Case 2

Among the edges joining \( K \) with \( C \) there are some edges from path system \( S \).
Since \( G \) is \((l + 3)\)-connected \( K \) may be joined with \( C \) by a family of \( K \)-ears and at list three edges which ends are not internal vertices of \( S \).

Hence \( G \) and \( S \) satisfies the following condition: if \( P : u_1 u_2 u_3 \subset S \) and \( d(u_1, G) \), \( d(u_2, G) \)
\[
\geq \frac{n+k}{2} \quad \text{then} \quad d(u_3, G) \geq \frac{n+k}{2} \quad \text{edges from} \ E(C, K) \cap E(S) \quad \text{may be as on Figure (3.2)}.
\]

![Figure (3.2)](image-url)
The graph $\tilde{G}$ has a minimal odd $M_S$-edge cut-set if there is a component $K$ of $GV \setminus C$ which is joined with cycle $C$ only by an odd number of edges from $E(S)$ or an odd number of edges from $E(G) \setminus E(S)$ with at least one end vertex in the set of internal vertices of path system $S$. In those cases the theorem is proved, so we may assume that $\tilde{G}$ has no minimal odd $M_S$-edge cut-set.

**Subcase 2.1.**

Among edges joining $K$ with $C$ we have an even number of edges from $E(S)$, say $s = 2r$, $(r \geq 1)$ which does not form any ear.

So we have vertices $x_1, \ldots, x_{2r} \in K$ and $y_1, \ldots, y_{2r} \in C$ such that $x_iy_i \in E(S)$, for $i = 1, \ldots, 2r$. We can assume that each edge $x_iy_i$ is in path of length 2 from path system $S$. Then we have vertices $x_i^+ \in V(K)$ such that $x_ix_i^+ \in E(S)$, for $i = 1, \ldots, 2r$.

Let $u, v \in V(C)$ be such that all edges from $E(C,K) \cap E(S)$ lying between $u$ and $v$ belong to some ears. In the cycle $C$ we have a path $W: uc_1 \ldots c_kv \subset C$. We shall define a new path $Q(u,v)$. If $u$ and $v$ are not in any ear. The path $Q(u,v)$ is a path joining $u$ with $v$ such that $E(W) \cap E(S) \subset E(Q(u,v))$ and $Q(u,v)$ contains all $c_i$ such that $c_i$ is not an internal vertex of a $K$-ear. In other words $Q(u,v)$ arises from $W$ by removing internal vertices of all $K$-ears. It is possible because if $c_i$ is an internal vertex of a $K$-ear then $c_i^+, c_i^- \in A$.

When $u$ is internal vertex of a $K$-ear, then we start the path $Q(u,v)$ from the first vertex $c_i$ which is not internal vertex of any $K$-ear. If $v$ is internal vertex of a $K$-ear, then we end the path $Q(u,v)$ from the last vertex $c_i$ which is not internal vertex of any $K$-ear.

The construction of $Q(u,v)$ is shown on figures (3.3) — (3.5).
First consider the path \( P_K \) containing only all \( K \)-ears. Now we can define the extension of the cycle \( C \) as follows (see Figure (3.6) (for \( r = 2 \))):

\[
C': \quad y_1 x_1 x_1^+ P_K x_2^+ x_2 Q(y_2, y_2^+) Q(y_3, y_3^+) \cdots y_{2r-1} \\
x_{2r-1} x_{2r-1}^+ v_1 \cdots v_s x_{2r}^+ x_{2r} Q(y_{2r}, y_{2r-1}^+) Q(y_{2r}, y_1),
\]

where \( x_{2r-1}^+ v_1 \cdots v_s x_{2r}^+ \) is a path containing all remaining vertices of \( K \) and edges of \( E(S) \cap E(K) \), this path exists because \( K \) is complete.

It is clear that the new cycle \( C' \) fulfils (3.3) — (3.5) and (3.7).
Subcase 2.2.

Among edges joining $K$ with $C$ we have an odd number of edges from $E(S)$, say $s = 2r - 1$, $(r \geq 1)$ which does not form any ear.

So we have vertices $x_1,...,x_{2r-1} \in V(K)$ and $y_1,...,y_{2r-1} \in V(C)$ such that $x_iy_i \in E(S)$. We can assume that each edge $x_iy_i$ is in path of length 2 from path system $S$. Then we have vertices $x_i^+ \in V(K)$ such that $x_i^+x_i \in E(S)$, for $i = 1,...,2r - 1$.

Since we have assumed that $\bar{G}$ has no minimal odd $M_S$-edge cut-set we have at least one edge say $xy$, $(x \in K, y \in C)$ such that $xy \notin E(S)$, $x$ and $y$ are not an internal vertices of $S$.

We shall consider four subcases according as $x$ or $y$ are extremities of an edge from the set $E(S)$.

Suppose that $y \notin \{y_1,...,y_{2r-1}\}$ and $x \notin \{x_1,...,x_{2r-1}\}$. In this case we have a vertex $y_{i0} \in V(C)$, $(i_0 \notin \{1,...,2r-1\})$ such that on the cycle $C$ the vertices are ordered as follows: $y_{i0}...y...y_{i0+1}$.

Consider a path $x_1...x_{i0}x_{i0+1}^+$ containing all vertices from the set $V(K) \setminus \{x_1,x_1^+,...,x_{i0},x_{i0}^+,x_{i0+2},x_{i0+2}^+,...,x_{2r-1},x_{2r-1}^+\}$ all $K$-ears and all edges from $E(S) \cap E(K)$.

If $yy \in E(S)$ consider the following cycle $C'$:

$$ C' : \quad y^{-}yx_1v_1...v_sx_{i0}^+x_{i0}+1x_{i0}+1Q(y_{i0}+1,y^+)Q(y_{i0}^+,y_{i0}+2)x_{i0}+2x_{i0}^+2x_{i0}^++3 \cdots Q(y_{i0}+3,y_{i0}^+)Q(y_{i0}^+,y_{i0}+4)Q(y_{i0}^+,y_{i0}+5)Q(y_{i0}^+,y^-); $$

satisfying properties: (3.2) — (3.5) and (3.7).

when $r = 1$ the edge $xy$ must be independent with all $x_iy_i$, so now we have $r \geq 2$.

Suppose that for $yy \notin \{y_1,...,y_{2r-1}\}$ and there is an $i_0 \notin \{1,...,2r-1\}$ such that $x = x_{i0}$. In this case $x_{i0}x_{i0}^+ \notin E(S)$.

If $yy \in E(S)$ then we define a new cycle $\bar{C}$ as follows:

$$ \bar{C} : \quad \bar{y}^{-}yx_{i0}Q(y_{i0},y^+)Q(y_{i0}^+,y^-). $$

and consider the complete graph $D$ obtained from $K$ by deletion of the vertex $x_{i0}$.

$D$ is a component of $G_{V(\bar{C})}$. Note that $\bar{C}$ and $D$ satisfies conditions (3.3) — (3.5) and (3.7). Since $r \geq 2$ $D$ is joined with $\bar{C}$ by an even number of edges from $E(S)$, which does not form any ear and then we can proceed as in subcase (2.1).

Suppose that for some $i_0,j_0 \in \{1,...,2r-1\}$ $x = x_{i0}$, $y = y_{j0}$ and $(i_0 \neq j_0)$.

First consider the case $r = 2$ and vertices $y_1, y_2, y_3$ are ordered in $C$ as follows: $y_1...y_2...y_3$.

We can assume that $y = y_1$, $x = x_3$ $(x_3x_3\notin E(S))$ and then consider the cycle:

$$ C' : \quad y_3x_3y_1v_1...v_sx_2Q(v_2,y_1^+)Q(y_1^-,y_3^+)Q(y_2^+,y_3), $$

where $x_1v_1...v_sx_2$ is a path containing all remaining vertices from $K$ all $K$-ears and all edges from $E(S) \cap E(K)$.

Again the cycle $C'$ has properties: (3.2) — (3.5) and (3.7).
when \( r > 2 \) we have \( y_l x_l \in E(S) \) and we assume that in the cycle \( C \) vertices are ordered as follows: \( y_0 \ldots y_l y_{l0} \). Now we can define a new cycle \( \tilde{C} \):

\[
\tilde{C}' : \quad y_{i0} x_{i0} y_{j0} x_{j0} x_l^+ x_l Q(y_l, y_{j0}^+) Q(y_{j0}^-, y_{i0}^+) Q(y_{l}^+, y_{i0}).
\]

and consider the complete graph \( D \) obtained from \( K \) by deletion of the vertices \( x_{i0}, x_l, x_{j0} \). \( D \) is a component of \( G_{V \setminus \tilde{C}} \). Note that \( \tilde{C} \) and \( D \) satisfies conditions (3.3) — (3.5) and (3.7). Since \( r > 2 \) \( D \) is joined with \( \tilde{C} \) by an even number of edges from \( E(S) \), which does not form any ear and a family of ears, so we can proceed as in subcase (2.1).

Subcase 2.3.

Among edges from \( E(S) \) joining \( K \) with \( C \) we have only edges which are forming \( K \)-ears.

Hence \( G \) is \( l + 3 \) connected we have also at least 3 edges from \( E(G) \setminus E(S) \) which ends are not internal vertices of \( S \).

This case is similar to the case 1. The only difference is fact that we have \( K \)-ears, but using paths \( Q(u,v) \) we can extend the cycle as in case 1.

In all cases we have extended the cycle \( C \), so the proof is complete.

4. CONCLUSIONS

The proof of Theorem 2.1 is an example of application of the closure technique. Note that the construction of the cycle \( C \) in the closure of the graph \( G \) is algorithmic but unfortunately it is possible that the cycle is using edges that does not belong to the initial graph.

Our result is an extension of Theorem 1.6.

References


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