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Solution and Generalized Ulam-Hyers Stability of a Reciprocal Type Functional Equation in Non-Archimedean Fields

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ABSTRACT

In this paper, we obtain the general solution of a reciprocal type functional equation of the type

$$f(x+y)=\frac{f\left(\frac{k_1x+k_2y}{k}\right)f\left(\frac{k_2x+k_1y}{k}\right)}{f\left(\frac{k_1x+k_2y}{k}\right)+f\left(\frac{k_2x+k_1y}{k}\right)}$$

and investigate its generalized Ulam-Hyers stability in non-Archimedean fields where $k>2$, k_1 and k_2 are positive integers with $k=k_1+k_2$ and $k_1 \neq k_2$. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

***Keywords*:** Reciprocal function; Reciprocal type functional equation; Generalized Hyers-Ulam stability

1. INTRODUCTION

A significant question concerning the theory of stability of functional equations was raised by S.M. Ulam [30] in 1940 in the University of Wisconsin. In 1941, D.H. Hyers [14] was the first person who presented an affirmative partial answer to the question of Ulam. In 1950, the theorem formulated by Hyers was generalized by T. Aoki [4] for additive mappings. In 1978, Th.M. Rassias [29] generalized Hyers' theorem which allows the Cauchy di_erence to be unbounded. In 1982, J.M. Rassias [21] gave a further generalization of the result of D.H. Hyers and proved theorem using weaker conditions controlled by a product of different powers of norms. In 1994, a generalization of Th.M. Rassias' theorem was obtained by P. Gavruta [12] who replaced the unbounded Cauchy difference by a general control function.

In 2008, J.M. Rassias et.al. [22] discussed the stability of quadratic functional equation

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2(m^2-2)f(x)-2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0$; $m \neq \pm 1$; $m \neq \sqrt{2}$ using mixed product-sum of powers of norms. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [7-9,11, 13,18,19,20,23]). Many research monographs are also available in functional equations, one can see ([1-3,10,15-17]).

Several mathematicians have remarked interesting applications of the Hyers-Ulam-Rassias stability theory to various mathematical problems. Stability theory is applied in fixed point theory to find the expression of the asymptotic derivative of a nonlinear operator.

The stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [31] used a stability property of the functional equation $f(x+y)+f(x-y)=2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

In 2010, K. Ravi and B.V. Senthil Kumar [24] obtained Ulam-Gavruta-Rassias stability for the reciprocal functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)} \qquad (1.1)$$

where $f: X \to \mathbb{R}$ is a mapping with X as space of non-zero real numbers. The reciprocal function $f(x)=\frac{c}{x}$ is a solution of the functional equation (1.1).

K. Ravi, J.M. Rassias and B.V. Senthil Kumar [25] discussed the Ulam stability for the reciprocal functional equation in several variables of the form

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{\substack{j=1,\\ j\neq i}}^{m} f(x_j)\right)\right]} \tag{1.2}$$

for arbitrary but fixed real numbers $(\alpha_1, \alpha_2, \dots, \alpha_m) \neq (0,0,\dots,0)$, so that $0 < \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m = \sum_{i=1}^{m} \alpha_i \neq 1$ and $f: X \to \mathbb{R}$ is a mapping with X as space of non-zero real numbers.

Later, J.M. Rassias et. al. ([26], [27]) introduced the Reciprocal Difference Functional equation (or RDF equation)

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1.3}$$

and the Reciprocal Adjoint Functional equation (or RAF equation)

$$f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)f(y)}{f(x)+f(y)} \tag{1.4}$$

and investigated the generalized Hyers-Ulam Rassias stability of the equations (1.3) and (1.4) using direct and fixed point methods.

A. Bodaghi and S.O. Kim [5] introduced and studied the Ulam-Gavruta-Rassias stabiility for the quadratic reciprocal functional mapping $f: X \to Y$ satisfying the Rassias quadratic reciprocal functional equation

$$f(2x+y) + f(2x-y) = \frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}. \tag{1.5}$$

The quadratic reciprocal function $f(x) = \frac{c}{x^2}$ is a solution of the functional equation (1.5).

Recently, A. Bodaghi and Y. Ebrahimdoost [6] generalized the equation (1.5) as

$$f\big((a+1)x+ay\big) + f\big((a+1)x-ay\big) = \frac{2f(x)f(y)[(a+1)^2 f(y)+a^2 f(x)]}{((a+1)^2 f(y)-a^2 f(x))^2} \tag{1.6}$$

where $a \in \mathbb{Z}$ with $a \neq 0$ and established the generalized Hyers-Ulam stability for the functional equation (1.6) in non-Archimedean fields.

K. Ravi et. al [27] investigated the generalized Hyers-Ulam stability of a reciprocal-quadratic functional equation of the form

$$r(x+2y) + r(2x+y) = \frac{r(x)r(y)\left[5r(x)+5r(y)+8\sqrt{r(x)r(y)}\right]}{\left[2r(x)+2r(y)+5\sqrt{r(x)r(y)}\right]^2} \tag{1.7}$$

in intuitionistic fuzzy normed spaces.

In this paper, we obtain the general solution of a reciprocal type functional of the type

$$f(x+y) = \frac{f\left(\frac{k_1 x + k_2 y}{k}\right) f\left(\frac{k_2 x + k_1 y}{k}\right)}{f\left(\frac{k_1 x + k_2 y}{k}\right) + f\left(\frac{k_2 x + k_1 y}{k}\right)} \tag{1.8}$$

and investigate the generalized Hyers-Ulam stability of the equation (1.8) in non-Archimedean fields where $k > 2$, k_1 and k_2 are positive integers with $k = k_1 + k_2$ and $k_1 \neq k_2$. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the equation (1.8).

In the above functional equation (1.8), if $(k_1, k_2) = (2,1)$ and $(k_1, k_2) = (3,2)$, then we obtain the following functional equations

$$f(x+y) = \frac{f\left(\frac{2x+y}{3}\right) f\left(\frac{x+2y}{3}\right)}{f\left(\frac{2x+y}{3}\right) + f\left(\frac{x+2y}{3}\right)}$$

and

$$f(x+y) = \frac{f\left(\frac{2x+y}{3}\right) f\left(\frac{x+2y}{3}\right)}{f\left(\frac{2x+y}{3}\right) + f\left(\frac{x+2y}{3}\right)}$$

respectively.

2. PRELIMINARIES

A non-Archimedean field is a field $\mathbb{A}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{A}$ into $[0, \infty)$ such that for all $r, s \in \mathbb{A}$,

(i) $|r| = 0$ if and only if $r = 0$
(ii) $|rs| = |r||s|$ and
(iii) $|r + s| \leq \max\{|r|, |s|\}$.

Clearly $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \in \mathbb{A}$ such that $|a_0| \neq 0,1$.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|r| = \begin{cases} 0 \text{ if } r = 0 \\ \frac{1}{r} \text{ if } r > 0 \\ -\frac{1}{4} \text{ if } r < 0 \end{cases}$$

for any $r \in \mathbb{A}$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n.

3. GENERAL SOLUTION OF EQUATION (1.8)

Theorem 3.1. *Let $f: \mathbb{R}^* \to \mathbb{R}$ be a function. Then f satisfies (1.1) if and only if f satisfies (1.8). Hence (1.8) is also a reciprocal mapping whose solution is $f(x) = \frac{c}{x}$.*

Proof. Let f satisfy (1.1). Replacing (x, y) by $\left(\frac{k_1x+k_2y}{k}, \frac{k_2x+k_1y}{k}\right)$ in (1.1), we arrive at (1.8). Conversely, suppose f satisfy (1.8). Replacing (x, y) by $\left(\frac{k_1x-k_2y}{k_1-k_2}, \frac{k_1y-k_2x}{k_1-k_2}\right)$ in (1.8), we obtain (1.1). This completes the proof of Theorem 3.1.

4. GENERALIZED ULAM-HYERS STABILITY OF EQUATION (1.8)

In the following theorems and corollaries, we assume that $\mathbb{A}$ and $\mathbb{B}$ be a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimedean field $\mathbb{A}$, we put $\mathbb{A}^* - \{0\}$. For convenience, let us define the difference operator $D_f: \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}$ by

$$D_f(x, y) = f(x + y) - \frac{f\left(\frac{k_1x+k_2y}{k}\right)f\left(\frac{k_2x+k_1y}{k}\right)}{f\left(\frac{k_1x+k_2y}{k}\right)+f\left(\frac{k_2x+k_1y}{k}\right)}$$

for all $x, y \in \mathbb{A}^*$ where $k > 2$, k_1 and k_2 are positive integers with $k = k_1 + k_2$ and $k_1 \neq k_2$.

Theorem 4.1. *Let $\phi: \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ be a function such that*

$$lim_{n\to\infty} \left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) = 0 \tag{4.1}$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f: \mathbb{A}^* \to \mathbb{B}$ is a mapping satisfying the inequality

$$\left|D_f(x, y)\right| \leq \phi(x, y) \tag{4.2}$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique reciprocal mapping $r: \mathbb{A}^* \to \mathbb{B}$ such that

$$|f(x) - r(x)| \leq max\left\{\left|\frac{1}{2}\right|^i \phi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) : i \in \mathbb{N} \cup \{0\}\right\} \tag{4.3}$$

for all $x \in \mathbb{A}^*$.

Proof. Replacing (x, y) by (x, x) in (4.2), we get

$$\left|f(2x) - \frac{1}{2}f(x)\right| \leq \phi(x, x) \tag{4.4}$$

for all $x \in \mathbb{A}^*$. Now, replacing x by $\frac{x}{2}$ in (4.4), we obtain

$$\left|f(x) - \frac{1}{2}f\left(\frac{x}{2}\right)\right| \leq \phi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{4.5}$$

for all $x \in \mathbb{A}^*$. Plugging x by $\frac{x}{2^n}$ in (4.5) and multiplying by $\left|\frac{1}{2}\right|^n$, we have

$$\left|\frac{1}{2^n}f\left(\frac{x}{2^n}\right) - \frac{1}{2^{n+1}}f\left(\frac{x}{2^{n+1}}\right)\right| \leq \left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) \tag{4.6}$$

for all $x \in \mathbb{A}^*$. Using (4.1), we find that the right-hand side of (4.6) tends to zero as $n \to \infty$.

Thus the sequence $\left\{\frac{1}{2^n}f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence. Completeness of the non-Archimedean space $\mathbb{B}$ allows us to assume that there exists a mapping r so that

$$r(x) = \lim_{n\to\infty} \frac{1}{2^n}f\left(\frac{x}{2^n}\right). \tag{4.7}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$\begin{aligned}\left|\frac{1}{2^n}f\left(\frac{x}{2^n}\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1}\left\{\frac{1}{2^{i+1}}f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i}f\left(\frac{x}{2^i}\right)\right\}\right| \\ &\leq max\left\{\left|\frac{1}{2^{i+1}}f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i}f\left(\frac{x}{2^i}\right)\right| : 0 \leq i < n\right\} \\ &\leq max\left\{\left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) : 0 \leq i < n\right\}. \end{aligned} \tag{4.8}$$

Applying (4.7) and letting $n \to \infty$, we find that inequality (4.3) holds. From (4.1), (4.2) and (4.7), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned}|D_r(x, y)| &= \lim_{n\to\infty}\left|\frac{1}{2}\right|^n \left|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right| \\ &\leq \lim_{n\to\infty}\left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0.\end{aligned}$$

Hence the mapping r satisfies (1.8). By Theorem 3.1, the mapping r is reciprocal. Now, let $R: \mathbb{A}^* \to \mathbb{B}$ be another reciprocal mapping satisfying (4.3).

Then we have

$$
\begin{aligned}
|r(x) - R(x)| &= \lim_{p\to\infty} \left|\frac{1}{2}\right|^p \left| r\left(\frac{x}{2^p}\right) - R\left(\frac{x}{2^p}\right)\right| \\
&\le \lim_{p\to\infty} \left|\frac{1}{2}\right|^p max\left\{\left| r\left(\frac{x}{2^p}\right) - f\left(\frac{x}{2^p}\right)\right|, \left| f\left(\frac{x}{2^p}\right) - R\left(\frac{x}{2^p}\right)\right|\right\} \\
&\le \lim_{p\to\infty} \lim_{q\to\infty} max\left\{ max\left\{\left|\frac{1}{2}\right|^{i+p} \phi\left(\frac{x}{2^{i+p+1}}, \frac{x}{2^{i+p+1}}\right) : p \le i \le q+p\right\}\right\} \\
&= 0
\end{aligned}
$$

for all $x \in \mathbb{A}^*$, proving that r is unique, which completes the proof.

Theorem 4.2. *Let* $\phi: \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ *be a function such that*

$$lim_{n\to\infty} |2|^n \phi(2^n x, 2^n y) = 0 \tag{4.9}$$

for all $x, y \in \mathbb{A}^*$. *Suppose that* $f: \mathbb{A}^* \to \mathbb{B}$ *is a mapping satisfying the inequality* (4.2) *for all* $x, y \in \mathbb{A}^*$. *Then there exists a unique reciprocal mapping* $r: \mathbb{A}^* \to \mathbb{B}$ *such that*

$$|f(x) - r(x)| \le max\left\{|2|^{i+1} \phi\left(2^i x, 2^i x\right) : i \in \mathbb{N} \cup \{0\}\right\} \tag{4.10}$$

for all $x \in \mathbb{A}^*$.

Proof. Replacing (x, y) by (x, x) in (4.2) and multiplying by $|2|$, we get

$$|2f(2x) - f(x)| \le |2|\, \phi(x, x) \tag{4.11}$$

for all $x \in \mathbb{A}^*$. Switching x to $2^n x$ in (4.11) and multiplying by $|2|^n$, we have

$$|2^n f(2^n x) - 2^{n+1} f(2^{n+1} x)| \le |2|^{n+1} \phi(2^n x, 2^n x) \tag{4.12}$$

for all $x \in \mathbb{A}^*$. As $n \to \infty$ in (4.12) and using (4.9), we see that the sequence $\{2^n f(2^n x)\}$ is a Cauchy sequence. Since $\mathbb{B}$ is complete, this Cauchy sequence converges to a mapping $r: \mathbb{A}^* \to \mathbb{B}$ defined by

$$r(x) = \lim_{n\to\infty} 2^n f(2^n x). \tag{4.13}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$
\begin{aligned}
|2^n f(2^n x) - f(x)| &= \left|\sum_{i=0}^{n-1} 2^{i+1} f\left(2^{i+1} x\right) - 2^i f\left(2^i x\right)\right| \\
&\le max\left\{\left|2^{i+1} f\left(2^{i+1} x\right) - 2^i f\left(2^i x\right)\right| : 0 \le i < n\right\} \\
&\le max\left\{|2|^{i+1} \phi\left(2^i x, 2^i x\right) : 0 \le i < n\right\}. \qquad (4.14)
\end{aligned}
$$

Applying (4.13) and letting $n \to \infty$, we find that the inequality (4.10) holds. From (4.9), (4.2) and (4.13), we have for all $x, y \in \mathbb{A}^*$

$$|D_r(x,y)| = \lim_{n\to\infty}|2|^n\left|D_f(2^n x, 2^n y)\right| \\ \leq \lim_{n\to\infty}|2|^n\phi(2^n x, 2^n y) = 0.$$

Hence the mapping r satisfies (1.8). By Theorem 3.1, the mapping r is reciprocal. Now, let $R: \mathbb{A}^* \to \mathbb{B}$ be another reciprocal mapping satisfying (4.10). Then we have

$$\begin{aligned} |R(x) - r(x)| &= \lim_{p\to\infty}|2|^p|R(2^p x) - r(2^p x)| \\ &\leq \lim_{p\to\infty}|2|^p max\{|R(2^p x) - f(2^p x)|, |f(2^p x) - r(2^p x)|\} \\ &\leq \lim_{p\to\infty}\lim_{q\to\infty} max\left\{max\{2^{i+p+1}\phi(2^{i+p}x, 2^{i+p}x): p \leq i \leq q+p\}\right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that r is unique.

Corollary 4.3. *For any fixed $k_1 \geq 0$ and $\alpha \neq -1$, if $f: \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$\left|D_f(x,y)\right| \leq k_1(|x|^\alpha + |y|^\alpha)$$

for all $x, y \in \mathbb{A}^*$, then there exists a unique reciprocal mapping $r: \mathbb{A}^* \to \mathbb{B}$ satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{2k_1}{|2|^\alpha}|x|^\alpha, for\ \alpha < -1 \\ 4k_1|x|^\alpha, for\ \alpha > -1 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The required results are obtained by choosing $\phi(x,y) = k_1(|x|^\alpha + |y|^\alpha)$, for all $x, y \in \mathbb{A}^*$ in Theorem 4.2 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$ and proceeding by similar arguments as in Theorems 4.1 and 4.2.

Corollary 4.4. *Let $f: \mathbb{A}^* \to \mathbb{B}$ be a mapping and let there exist real numbers a, b such that $\alpha = a + b \neq -1$. Let there exists $k_2 \geq 0$ such that*

$$\left|D_f(x,y)\right| \leq k_2|x|^a|y|^b$$

for all $x, y \in \mathbb{A}^*$ Then there exists a unique reciprocal mapping $r: \mathbb{A}^* \to \mathbb{B}$ satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{k_2}{|2|^\alpha}|x|^\alpha, for\ \alpha < -1 \\ 2k_2|x|^\alpha, for\ \alpha > -1 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. Considering $\phi(x,y) = k_2|x|^a|y|^b$, for all $x, y \in \mathbb{A}^*$ in Theorem 4.2 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$, the proof of the corollary is complete.

Corollary 4.5. *Let $k_3 \geq 0$ and $\alpha \neq -1$ be real numbers, and $f: \mathbb{A}^* \to \mathbb{B}$ be a mapping satisfying the functional inequality*

$$\left|D_f(x,y)\right| \leq k_3\left(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + (|x|^\alpha + |y|^\alpha)\right)$$

for all $x, y \in \mathbb{A}^*$ Then there exists a unique reciprocal mapping $r: \mathbb{A}^* \to \mathbb{B}$ satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{3k_3}{|2|^\alpha}|x|^\alpha, for\ \alpha < -1 \\ 6k_3|x|^\alpha, for\ \alpha > -1 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The proof follows immediately by taking $\phi(x,y) = k_3\left(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + (|x|^\alpha + |y|^\alpha)\right)$, for all $x, y \in \mathbb{A}^*$ in Theorem 4.2 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$.

5. CONCLUSION

Thus, the generalized Ulam-Hyers stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by mixed product-sum of powers of norms for the reciprocal functional equation (1.8) hold good in non-Archimedean fields. Also the stability results obtained in Corollary 4.4 are better approximation than the stability results obtained in Corollary 4.5.

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